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MODERN GEOMETRY

THE STRAIGHT LINE AND CIRCLE

BY

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PREFACE

During the time that has elapsed since the publication in 1909 of *A Course of Plane Geometry for Advanced Students*, Part I., the author has received many friendly suggestions from those who have employed it as a text-book. Advantage is now taken of the need for a new edition to make a thorough revision and rewrite various portions of the book, in the light of the experience thus gained: in fact, so extensive are the changes, that it seemed desirable to select even a fresh title.

Some new material has been added, the old material has been re-arranged and occasionally condensed, but it is hoped that clearness has been nowhere sacrificed to brevity, and that the collection of riders, to which the author attaches great importance, has benefited by the omissions and additions which have been made.

C. V. D.

June 1919.

CONTENTS

CHAPTER I.

SIMILAR FIGURES.

PAGE

Similar Polygons 1

Homothetic Circles and Centres of Similitude . . 3

Pantograph 5

Areas 7

Constructions 8

CHAPTER II.

RATIO AND RECTANGLE PROPERTIES.

Ratio of Triangles 10

Transversal Property 13

Apollonius' Circle 16

Ptolemy's Theorem 17

CHAPTER III.

PROPERTIES OF THE TRIANGLE.

Notation 19

Circumcentre, Orthocentre, Centroid, Incentre, Excentres 20

Nine-Point Circle 27

PAGE

THE EULER LINE 28

PROPERTIES OF O, I, H 30

CHAPTER IV.

CONCURRENCY AND COLLINEARITY.

SENSE 37

THE POINT AT INFINITY ON A STRAIGHT LINE . . . 38

CEVA'S THEOREM 40

MENELAUS' THEOREM 42

ANOTHER THEOREM ON CONCURRENCY 45

SIMSON'S LINE 46

CHAPTER V.

VECTOR GEOMETRY AND STATICAL APPLICATIONS.

DEFINITIONS: VECTOR ADDITION 49

MEAN CENTRE 51

VECTOR MULTIPLICATION 53

GRAPHICAL CONSTRUCTION 55

INTERPRETATION OF $(r, \theta)=[x, y]$ 56

DEMOIVRE'S THEOREM 57

CENTRE OF GRAVITY 58

APOLLONIUS' THEOREM AND EXTENSIONS 61

CHAPTER VI.

HARMONIC RANGES AND PENCILS.

METRICAL PROPERTIES 65

PENCIL PROPERTIES 66

CROSS RATIO 73

CHAPTER VII.

THE QUADRANGLE AND QUADRILATERAL.

PAGE
The Principle of Duality - - - - - - 77
Harmonic Theory - - - - - - - - 80
Mid Points of Diagonals Collinear - - - - 85

CHAPTER VIII.

ORTHOGONAL CIRCLES.

Tests for Orthogonal Intersection - - - - 88
Harmonic Properties - - - - - - - 91

CHAPTER IX.

POLES AND POLARS.

Definitions - - - - - - - - - 93
Analytical Methods - - - - - - - 94
Conjugate Properties - - - - - - - 95
Harmonic Properties - - - - - - - 98
The Quadrangle and Quadrilateral - - - - 102

CHAPTER X.

INVERSION.

Definitions - - - - - - - - - 105
Inverse of Straight Line and Circle - - - - 106
Angle between Curves - - - - - - - 108
Metrical Properties - - - - - - - 112
Two Circles inverted into each other - - - - 114
Analytical Methods - - - - - - - 118

CHAPTER XI.

COAXAL CIRCLES.

PAGE
ANALYTICAL METHODS 121
RADICAL AXIS 124
COAXAL SYSTEM AND LIMITING POINTS 126
POLAR PROPERTY 131
APPLICATION TO QUADRILATERAL 134
CIRCLES OF SIMILITUDE 136
METRICAL TANGENT PROPERTIES 138
APPLICATION OF INVERSION 142

INDEX 144

MODERN GEOMETRY.

CHAPTER I.

SIMILAR FIGURES.

Definition.

Polygons which are equiangular and have their corresponding sides proportional are called *similar*.

If, in addition, their corresponding sides are parallel, they are said to be *similarly situated* or *homothetic*.

THEOREM 1.

If O is any fixed point and ABCD...X any polygon, and if points A′, B′, C′, ... X′ are taken on OA, OB, OC, ... OX (or those lines

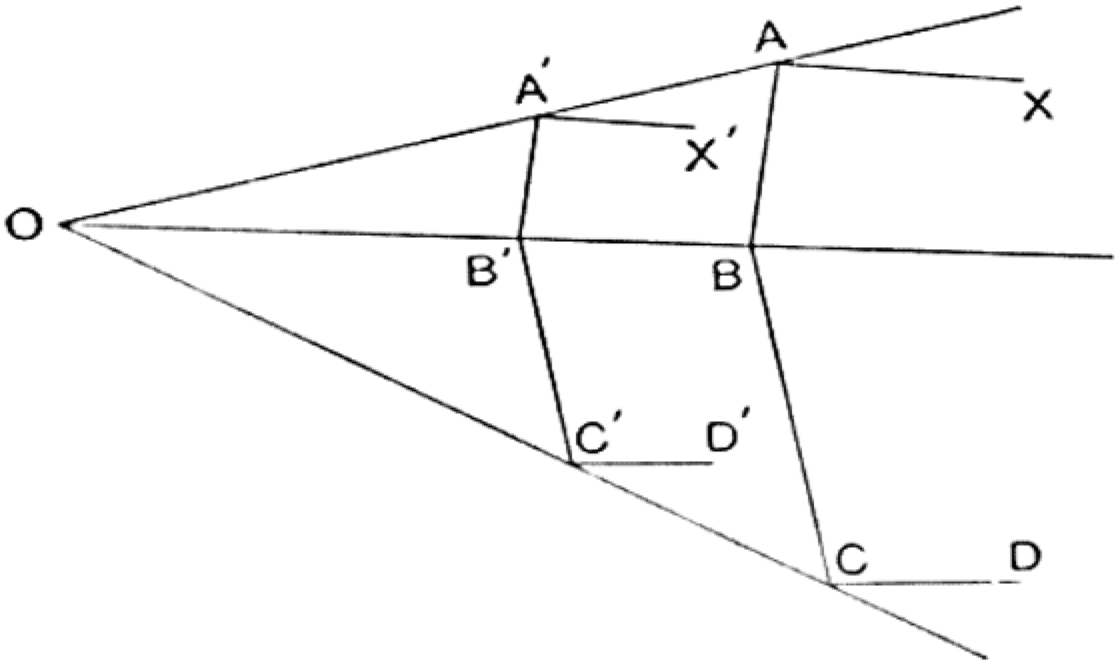

FIG. 1.

produced either way) such that $\frac{OA'}{OA}=\frac{OB'}{OB}=\ldots=\frac{OX'}{OX}=k$, then the polygons ABCD...X, A′B′C′D′...X′ are homothetic.

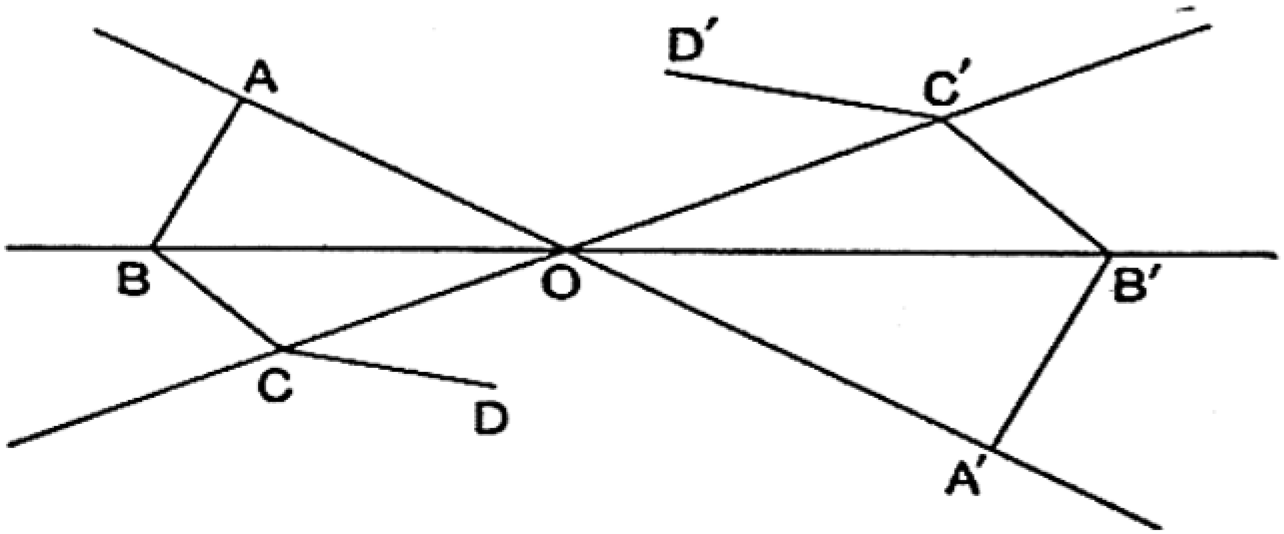

FIG. 2.

Since $\frac{OA'}{OA}=\frac{OB'}{OB}$, AB is parallel to A′B′, and the triangles OAB, OA′B′ are similar.

$$\therefore \frac{A'B'}{AB}=\frac{OA'}{OA}=k.$$

∴ corresponding sides are proportional and parallel.

∴ the polygons are homothetic. Q.E.D.

Definition.

If O is a fixed point and P is a variable point on a fixed curve S, and if P′ is a point on OP such that $\frac{OP'}{OP}=k$ (a constant), then the locus of P′ is a curve S′, which is said to be *homothetic* to S; and P, P′ are called *corresponding points.*

O is called the *centre of similitude* of the two figures.

If P and P′ lie on the same side of O, the figures are said to be *directly homothetic* w.r.t.* O, and O is called the *external* centre of similitude (see Fig. 1).

If P and P′ lie on opposite sides of O, the figures are said to be inversely homothetic w.r.t. O, and O is called the *internal* centre of similitude (see Fig. 2).

The parallel lines AB, A′B′ in Fig. 1 are said to be drawn in the *same sense.*

The parallel lines AB, A′B′ in Fig. 2 are said to be drawn in *opposite senses.*

---

* w.r.t. will be used as an abbreviation for "with respect to."

THEOREM 2.

A, B are the centres of any two circles of radii a, b; AB is divided externally at O and internally at O_1 in the ratio of the radii

$$\left[i.e.\ \frac{AO}{BO}=\frac{AO_1}{O_1B}=\frac{a}{b}\right]:$$

then the circles are directly homothetic w.r.t. O and inversely homothetic w.r.t. O_1, and corresponding points lie on the extremities of parallel radii.

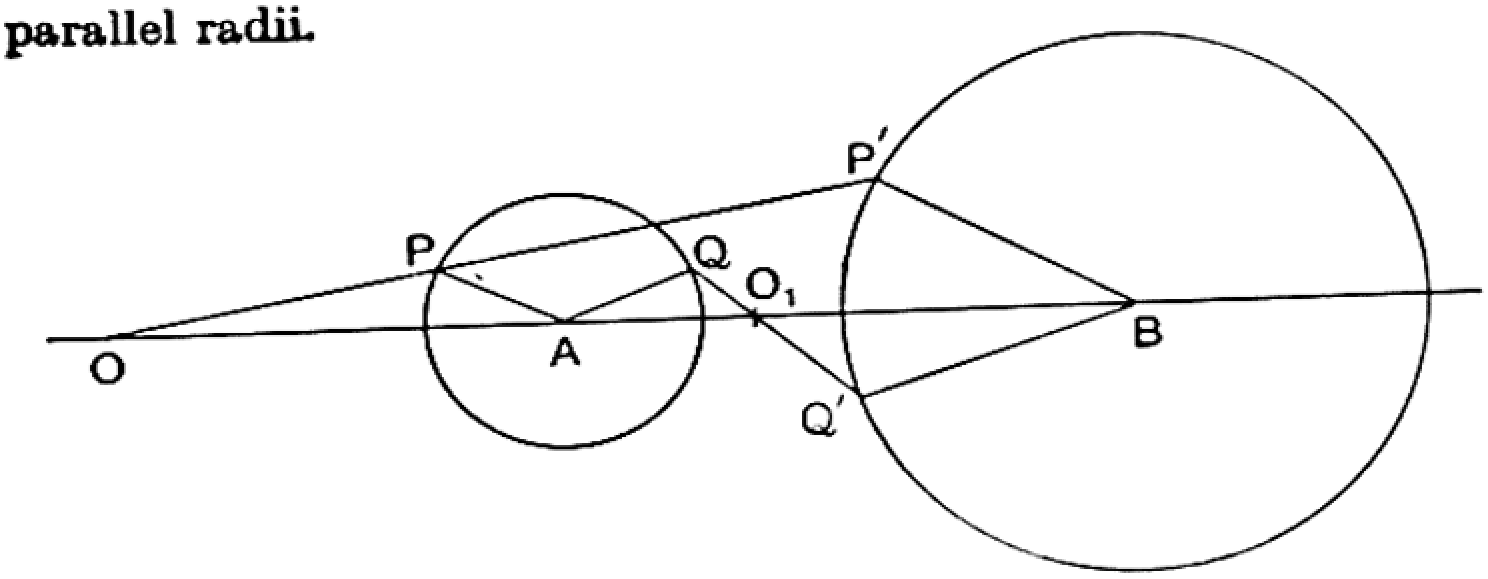

FIG. 3.

P is any point on the circle, centre A.

Produce OP to P′, so that $\frac{OP'}{OP}=\frac{b}{a}$. Join P′B.

Since $\frac{OP'}{OP}=\frac{b}{a}=\frac{OB}{OA}$, BP′ is parallel to AP.

$\therefore$ by similar triangles $\frac{BP'}{AP}=\frac{OB}{OA}=\frac{b}{a}$.

But $AP=a$; $\therefore$ $BP'=b$.

$\therefore$ P′ lies on the circle, centre B, radius b.

$\therefore$ the circles are directly homothetic w.r.t. O; and the radii through the corresponding points P, P′ are parallel.

Also, O is the external centre of similitude of the two circles.

Similarly, the circles are inversely homothetic w.r.t. O_1, and the radii through the corresponding points Q, Q′ (see Fig. 3) are parallel.

Also O_1 is the internal centre of similitude of the two circles.

Q.E.D.

1. If two polygons are homothetic, prove that the lines joining corresponding vertices are concurrent.

2. (1) If P, P′ and Q, Q′ are two pairs of corresponding points, prove that PQ is parallel to P′Q′.

(2) If P and P′ are corresponding points on two homothetic curves, prove that the tangents at P and P′ are parallel.

3. If two curves are homothetic w.r.t. O, and if OP is a tangent to one curve, prove that it is also a tangent to the other.

4. If two triangles are homothetic, prove that (1) the centres of their circumscribing circles, (2) their orthocentres are corresponding points.

5. Two plans of the same estate, on different scales, are placed one wholly on the other with their north-south lines parallel; prove that there is one point of contact between the maps at which the same place is represented in each map.

6. O is any fixed point: P is a variable point on a fixed circle: P′ is a point on OP, such that $\frac{OP'}{OP}$ is constant; prove that the locus of P′ is a circle.

7. Prove that the common tangents to two circles pass through one or other of the centres of similitude.

8. A is a fixed point: P is a variable point on a fixed circle, centre C; the line bisecting the angle ACP meets AP at Q. Find the locus of Q.

9. A line OHK is drawn through the centre of similitude of two circles to cut them at H, K; prove that the tangents at H, K are parallel.

10. Two circles intersect at C, D, and their common tangents intersect at T; CP, CQ are the tangents at C to the two circles; prove that CT bisects the angle PCQ.

11. Where is the internal centre of similitude of two circles which touch each other? Where is the external centre of similitude of two equal circles? Where are the centres of similitude of a circle and a straight line?

12. If S and S′ are two curves, each homothetic w.r.t. a third curve Σ (w.r.t. different centres), prove that S is homothetic to S′.

13. Two fixed circles touch each other at A; a variable line through A cuts the circles at P, Q; prove that $\frac{AP}{AQ}$ is constant.

14. A′, B′, C′ are the mid-points of the sides BC, CA, AB of a triangle ABC; G, H, O are the centroid, orthocentre, circumcentre of the triangle ABC; prove that (i) G is the centre of similitude of the triangles ABC, A′B′C′, (ii) O is the orthocentre of the triangle A′B′C′, and hence that G, H, O are collinear. What is the value of the ratio $\frac{GH}{GO}$?

15. P is a variable point on a fixed line AB; AC is another fixed line: a circle is drawn to touch AB, AC and the circle whose diameter is AP. Prove that the locus of the point of contact of the two circles is a straight line through A.

16. The centres of two circles of radii a, b are at distance d apart. Calculate the distance between the two centres of similitude.

17. A′, B′, C′ are the mid-points of the sides BC, CA, AB of a triangle ABC; prove the tangent at A to the circle ABC is parallel to the tangent at A′ to the circle A′B′C′.

18. P, Q are any two points on a circle; P′, Q′ are the corresponding points on a second circle w.r.t. a centre of similitude O; OP, OQ meet the first circle again at X, Y and the second circle at X′, Y′; prove that

(i) PQ is parallel to P′Q′, (ii) PQY′X′ is a cyclic quadrilateral,

(iii) OP . OX′ = OP′ . OX = OQ . OY′ = OQ′ . OY.

19. O is the external centre of similitude of two circles; OTT′ is an exterior common tangent; Q, Q′ are a pair of corresponding points w.r.t. O; prove that QT, Q′T′ are parallel.

20. Prove that a variable line through a centre of similitude of two fixed circles cuts off from the circles chords whose lengths are in a constant ratio, and similar segments.

21. OPQ is the external common tangent to two circles which intersect at A, B; if O is a centre of similitude, prove that OP . OQ = OA².

22. Two spheres, radii a, b $(a > b)$, are glued together at a point P. The solid so formed is placed on a horizontal table and is made to roll, without slipping, on the table; prove that P describes a circle of radius r given by $r^2(a^2-b^2)^2=16a^3b^3$.

23. The in-circle of the triangle ABC touches BC at X, and the circle escribed to BC touches BC at X′; if AX′ cuts the in-circle at F, G, prove that either FX or GX is a diameter of the in-circle.

24. Two circles APQ, AHK cut at A; PH, QK are their common tangents; prove that the circles APH, AQK touch each other.

25. If a circle S touches the circumcircle of the triangle ABC at P, prove that the tangents to S from A, B, C are in the ratio AP : BP : CP. What does this result become when the radius of S increases without limit?

The description of a figure, homothetic to a given figure, is sometimes effected by means of an instrument, called a *pantograph*. This might be employed, for example, when increasing or reducing the scale of a map or plan.

In the figure, rods are represented by straight lines. Two equal rods OA, AP are jointed together at A and the end-point O is fixed but OA can turn freely about O. Two other rods P′BH, P′CK are jointed together at P′ and are pinned to OA, AP respectively at B, C, so that BP′ = BO and BP′CA is a parallelogram.

If P is made to trace out any curve S, then P′ will trace out a homothetic curve S′.

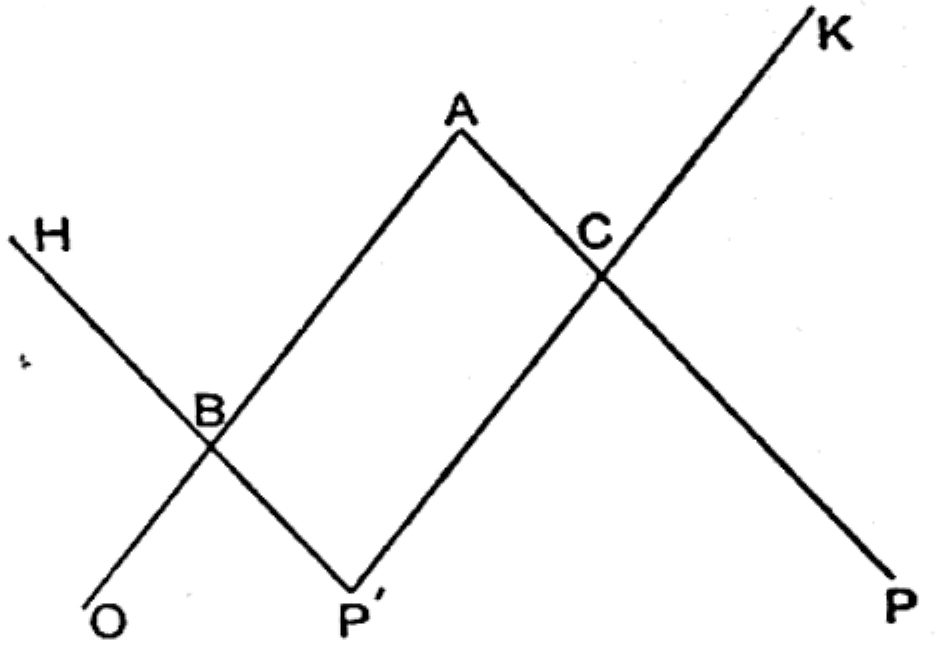

FIG. 4.

26. In Fig. 4, prove that (i) O, P′, P are collinear, (ii) $\frac{OP'}{OP}=\frac{OB}{OA}$.

THEOREM 3.

A triangle is given in species (*i.e.* its angles are given), one vertex is fixed, another lies on a given circle, then the locus of the third vertex is a circle.

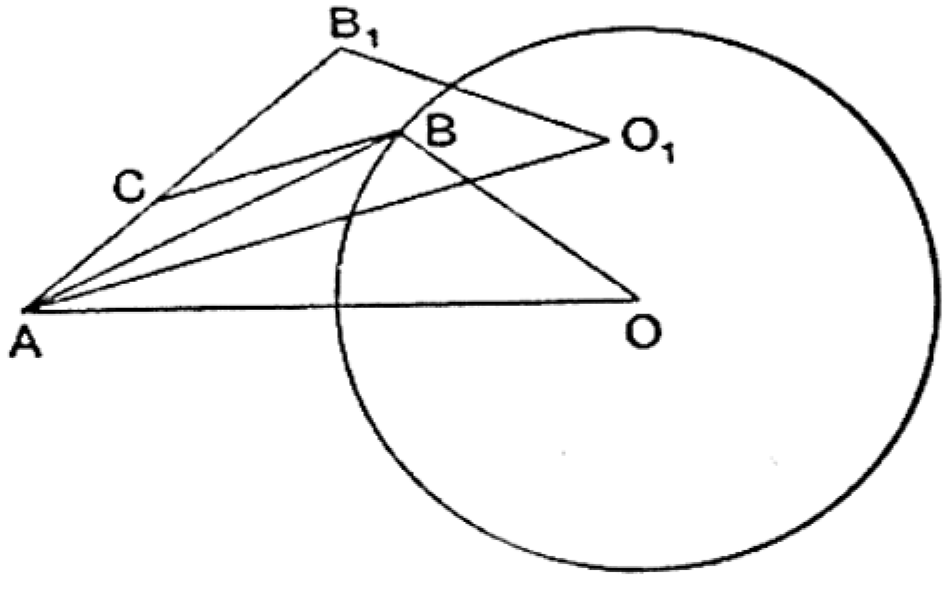

FIG. 5.

ABC is one position of the triangle ; A is fixed, B lies on a given circle, centre O.

On AC take a point B_1, such that $AB_1=AB$.

On AB_1 describe a triangle AO_1B_1 directly similar, and ∴ congruent to the triangle AOB.

Since $O_1\hat{A}B_1 = O\hat{A}B$, $O_1\hat{A}O = B_1AB =$ constant.

Also $AO_1 = AO =$ constant.

$\therefore$ O_1 is a fixed point.

But $O_1B_1 = OB =$ constant.

$\therefore$ B_1 describes a circle.

Now $\frac{AC}{AB_1} = \frac{AC}{AB} =$ constant, since the triangle ABC is of given shape.

$\therefore$ C traces out a curve similar to the locus of B_1, *i.e.* a circle.

Q.E.D.

Notice the connection between the locus of B and the locus of C. If the circle on which B moves is rigidly rotated about A through the constant angle BAC, its new position is the locus of B_1; if now this circle is uniformly contracted towards A, being altered in the constant ratio $\frac{AC}{AB}$, the new position of this circle is the locus of C.

THEOREM 4.

If the line CD is a mean proportional between the lines AB, EF, and if S, T are the areas of any two similar figures of which AB, CD are corresponding sides, then $\frac{S}{T} = \frac{AB}{EF}$.

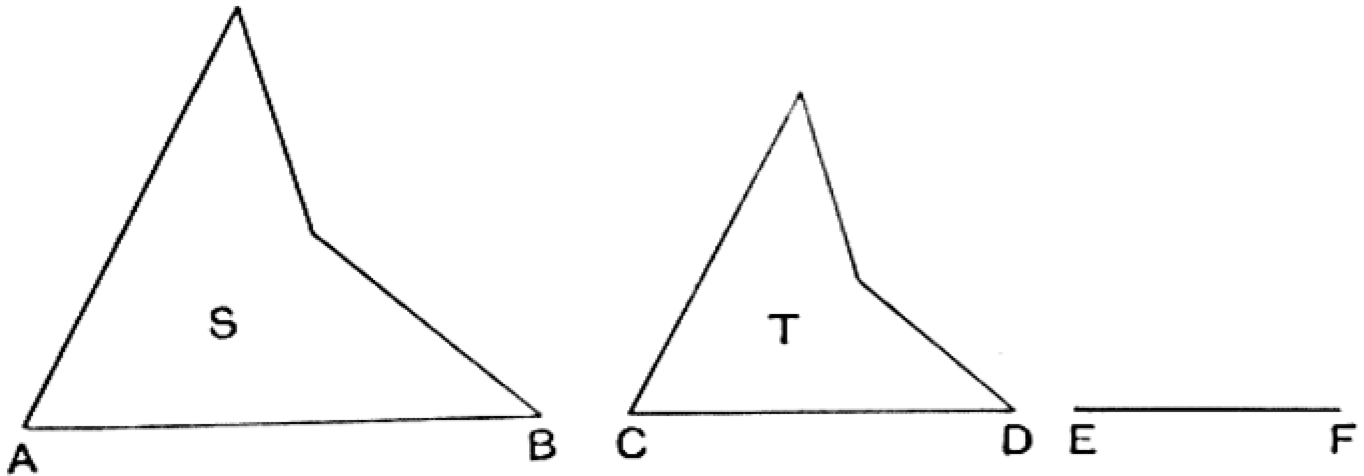

FIG. 6.

$$\frac{S}{T} = \frac{AB^2}{CD^2}; \quad \text{but } CD^2 = AB \,.\, EF.$$

$$\therefore \frac{S}{T} = \frac{AB^2}{AB \,.\, EF} = \frac{AB}{EF}.$$

Q.E.D

This Theorem is of use in various constructions (*e.g.* Ex. 44, 45). The following example illustrates another type of construction.

Example. To inscribe a square in any given triangle ABC.

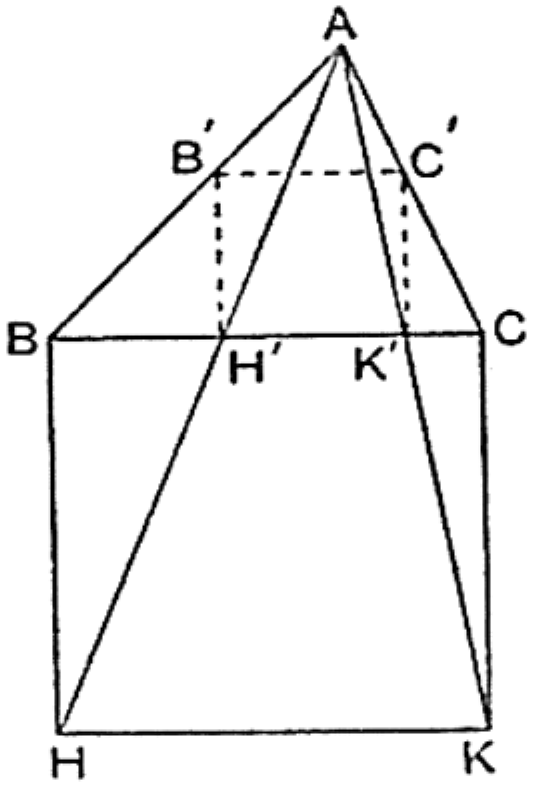

FIG. 7.

On BC describe a square BHKC externally to the triangle.

Join AH, AK, and let them cut BC at H′, K′.

Draw H′B′, K′C′ perpendicular to BC to cut AB, AC at B′, C′.

Then B′H′K′C′ is the required square.

To prove this, regard A as a centre of similitude.

27. A triangle ABC is given in species; A is fixed, B moves on a fixed straight line; find the locus of C.

28. A, B are two fixed points on a fixed circle; P is a variable point on the circle; Q is a point on BP, such that $\frac{BQ}{AP}$ is constant; find the locus of Q.

29. APQ is a variable triangle; A is fixed, P moves on a fixed line CD; if AP meets a fixed line parallel to CD at R, and if PQ = AR and if the angle APQ is constant, prove that the locus of Q is a straight line.

30. Two fixed circles intersect at A, B: a variable line through A cuts the circles at P, Q; prove that the locus of a point dividing PQ in a constant ratio is a circle. [Consider rotation and shrinking about B.]

31. The sides QR, RP, PQ of a variable triangle fixed in species pass through the fixed points A, B, C; prove that the locus of the orthocentre of the triangle PQR is a circle. [If the circles ABR, ACQ cut at E, prove E is a fixed point, and consider rotation and shrinking about E.]

32. B, B′ are two points on two lines AB, AB′. O is a point such that, by a rotation about O, the line AB and the point B can be made

homothetic to the line AB′ and the point B′; prove that the locus of O is the circle ABB′.

33. In a given triangle, inscribe a triangle having its sides parallel to three fixed lines.

34. Given a triangle ABC, construct a square such that two vertices lie on BA produced, CA produced and the opposite side along BC.

35. In a given triangle inscribe an equilateral triangle with one side parallel to a given line.

36. OB, OC are two given radii of a circle. Describe a square so that two of its vertices lie on OB, OC and the other two on the arc BC.

37. ABCD is a quadrilateral such that $BA = BC$ and $DA = DC$: inscribe a square in the quadrilateral.

38. ABCD is a square: construct a semicircle to touch BA, BC and to have the ends of its base on DA, DC.

39. Two given lines intersect at an inaccessible point H; A is a given point; construct the line AH.

40. Construct a circle to touch two given lines and to pass through a given point.

41. Inscribe in a given triangle a rectangle similar to a given rectangle.

42. Construct a square ABCD such that $AC - AB$ is of given length.

43. Construct a triangle similar to a given triangle, and such that its vertices lie on three given parallel straight lines.

44. Divide a triangle by a line parallel to one side into parts whose areas are in a given ratio.

45. Construct a triangle similar to a given triangle and equal in area to another given triangle.

46. A tangent at a point P of a circle meets a chord AB, when produced, at T; prove that $\frac{PA^2}{PB^2} = \frac{TA}{TB}$.

47. ABC is a triangle such that $AB = AC = 2BC$; D is a point in AC such that $D\hat{B}C = D\hat{A}B$; find the ratio of the areas of the triangles ABC, DBC.

CHAPTER II.

RATIO AND RECTANGLE PROPERTIES.

THEOREM 5.

Two triangles ABC, ABD have a common base AB; the line joining their vertices cuts AB at X; then

$$\frac{\triangle ACB}{\triangle ADB}=\frac{CX}{DX}.$$

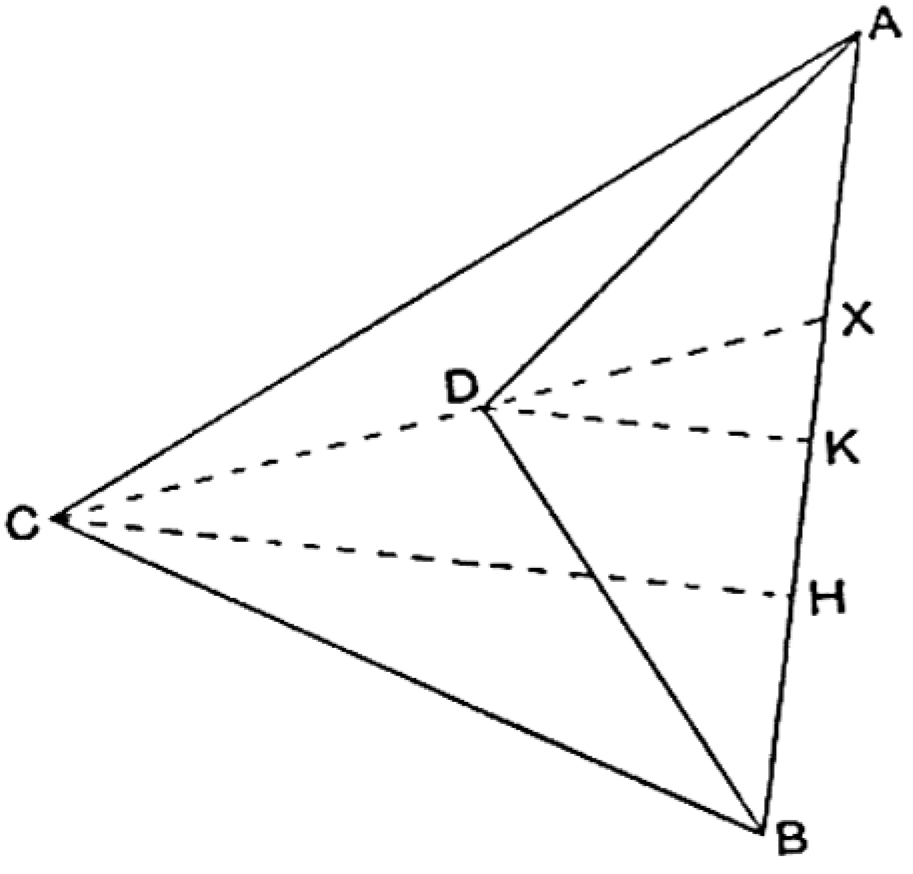

FIG. 8.

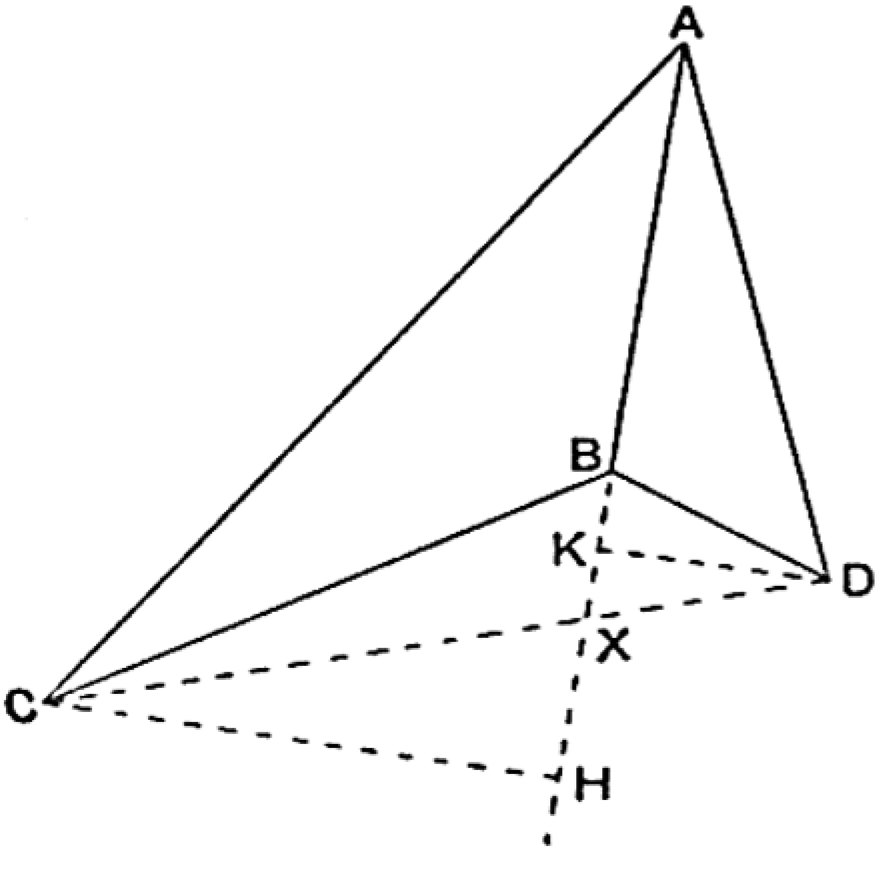

FIG. 9.

Drop perpendiculars CH, DK to AB.

$$\frac{\triangle ACB}{\triangle ADB}=\frac{\frac{1}{2}AB \cdot CH}{\frac{1}{2}AB \cdot DK}=\frac{CH}{DK}$$

$$=\frac{CX}{DX} \text{ by similar triangles.}$$

Q.E.D.

THEOREM 6.

If AD is an altitude of the triangle ABC, and if R is the radius of the circumcircle then AB . AC = 2R . AD.

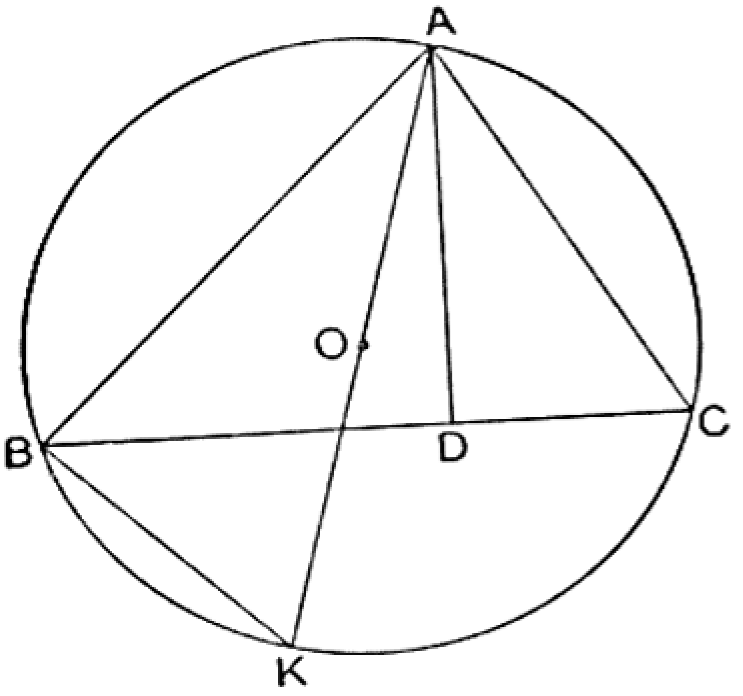

FIG. 10.

Draw the diameter AK of the circumcircle ; join BK.

The triangles ADC, ABK are similar,

for $\hat{ADC} = 90° = \hat{ABK}$, since AK is a diameter.

$\hat{ACD} = \hat{AKB}$ in same segment.

$$\therefore \frac{AB}{AK} = \frac{AD}{AC}$$

$\therefore$ **AB . AC = AK . AD = 2R . AD.** Q.E.D.

THEOREM 7.

If P is any point on a chord AB of a circle, centre O, radius R, then $AP \cdot PB = R^2 - OP^2$.

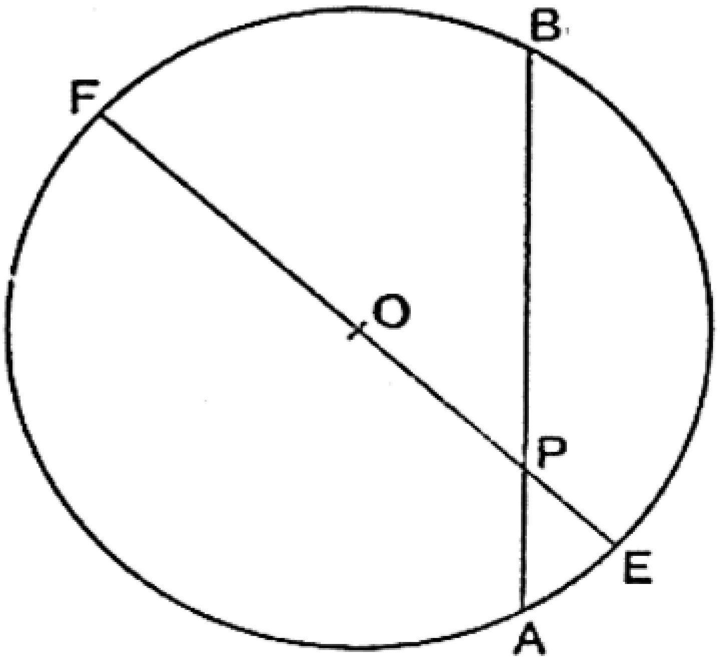

FIG. 11.

Join OP and produce it each way to cut the circle in E, F.

Then $AP \cdot PB = EP \cdot PF$

$$= (OE - OP)(OF + OP) = (R - OP)(R + OP)$$

$$= R^2 - OP^2.$$ Q.E.D.

Note that the theorem still holds if P lies on AB produced. In this case, it may be written $PA \cdot PB = OP^2 - R^2$.

THEOREM 8.

If any straight line is drawn cutting three concurrent lines OA, OB, OP at A, B, P, then

$$\frac{AP}{PB} = \frac{AO \sin AOP}{BO \sin POB}.$$

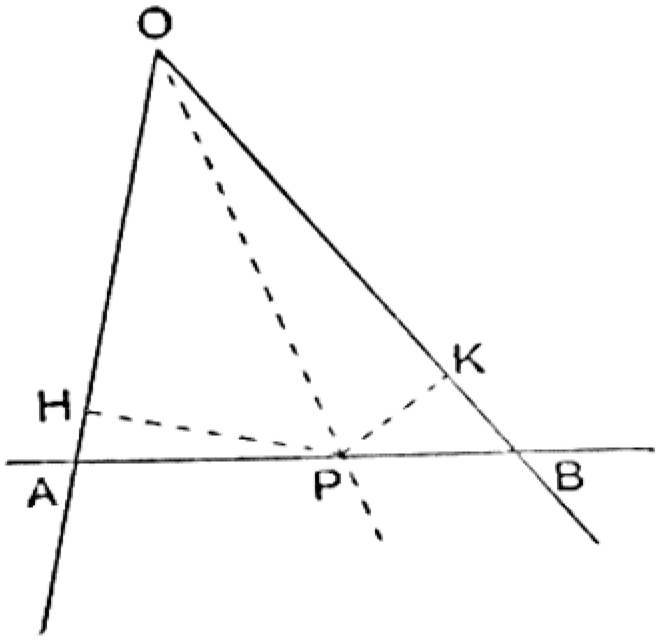

FIG. 12.

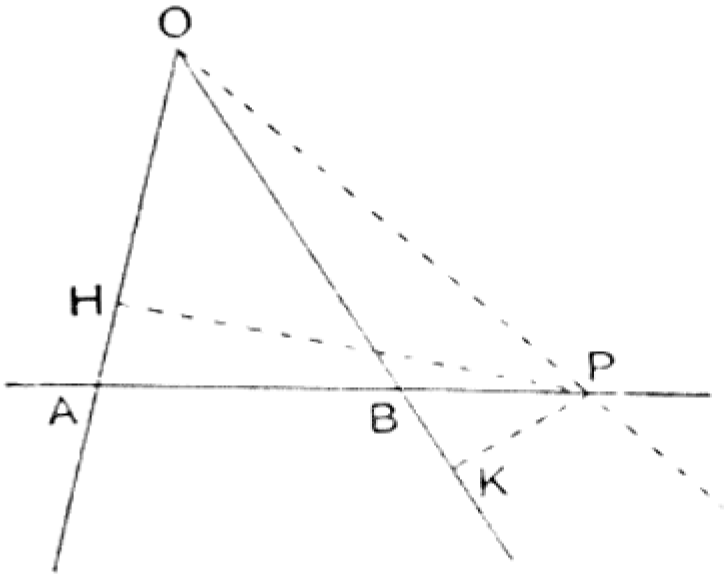

FIG. 13.

Draw PH, PK perpendicular to OA, OB.

$$\frac{AP}{PB} = \frac{\triangle APO}{\triangle PBO} = \frac{\frac{1}{2}PH \,.\, AO}{\frac{1}{2}PK \,.\, BO} = \frac{PO \sin AOP \,.\, AO}{PO \sin POB \,.\, BO} = \frac{AO \sin AOP}{BO \sin BOP}.$$

Q.E.D.

Note the two cases, Fig. 12 and Fig. 13.

The formula $\Delta = \frac{1}{2}bc \sin A$ gives an easy proof.

Definition.

Any line cutting a system of straight lines is called a *transversal* of those lines.

1. If in the triangles ABC, XYZ, the angles BAC, YXZ are equal or supplementary, prove that $\frac{\triangle ABC}{\triangle XYZ} = \frac{AB \,.\, AC}{XY \,.\, XZ}$.

2. In the triangles, ABC PQR,

$$A\hat{B}C = P\hat{Q}R \quad \text{and if } A\hat{C}B + P\hat{R}Q = 180^\circ,$$

prove that $$\frac{AC}{PR} = \frac{AB}{PQ}.$$

3. O is any point inside the triangle ABC; AO, BO, CO meet BC, CA, AP at P, Q, R; prove that

$$\text{(i)}\ \frac{BP}{PC}\times\frac{CQ}{QA}\times\frac{AR}{RB}=1,\quad \text{(ii)}\ \frac{OP}{AP}+\frac{OQ}{BQ}+\frac{OR}{CR}=1.$$

4. O is any point on the line bisecting the angle BAC; prove that

$$\frac{\triangle BAO}{\triangle CAO}=\frac{BA}{AC}.$$

5. Find a point O inside the triangle ABC such that

$$\triangle AOB : \triangle BOC : \triangle COA = p : q : r,$$

where p, q, r are given constants.

6. Points P, Q, R are taken on the sides BC, CA, AB of the triangle ABC so that $BP : PC = CQ : QA = AR : RB = \frac{1}{2}$; prove that

$$\triangle PQR = \tfrac{1}{3}\triangle ABC.$$

7. Points P, Q are taken on the sides BC, CA of the triangle ABC so that $BP = 2PC$, $CQ = 3QA$; AP meets BQ at O; calculate ratio area POQC : △ ABC.

8. The diagonals AC, BD of a cyclic quadrilateral ABCD meet at O; prove that $\dfrac{AB\,.\,BC}{AD\,.\,DC}=\dfrac{BO}{OD}$.

9. Two circles touch internally at O; a straight line ABCD cuts the outer at A, D and the inner at B, C; prove that

$$AB : CD = OA\,.\,OB : OC\,.\,OD.$$

10. If $a=40$, $b=68$, $c=84$, prove that $R=42{\cdot}5$.

11. P is a point on the circle circumscribing a quadrilateral ABCD; perpendiculars PL, PM, PN, PR, PS, PT are let fall to AB, BC, CD, DA, AC, BD; prove that $PL\,.\,PN = PM\,.\,PR = PS\,.\,PT$.

12. The tangents at the extremities of a chord AB of a circle meet at C. From a point P on the circle, perpendiculars PL, PM, PN are drawn to AB, BC, CA; prove that $PM\,.\,PN = PL^2$.

13. OP, OQ are two variable chords of a fixed circle. O is a fixed point; if OP . OQ is constant, find the envelope of PQ.

14. P is any point on the circle circumscribing a hexagon ABCDEF; PL_1, PL_2, PL_3, PL_4, PL_5, PL_6 are the perpendiculars from P to consecutive sides; prove that $PL_1\,.\,PL_3\,.\,PL_5 = PL_2\,.\,PL_4\,.\,PL_6$.

What theorem is obtained by making A, C, E coincide respectively with B, D, F?

15. Generalise Ex. 14 so as to obtain a theorem for a $2n$-sided figure inscribed in a circle.

16. P, Q, R are any points on the sides BC, CA, AB of a triangle: O is any point inside the triangle: AP′, BQ′, CR′ are drawn parallel to OP, OQ, OR to meet BC, CA, AB at P′, Q′, R′; prove that

$$\frac{OP}{AP'}+\frac{OQ}{BQ'}+\frac{OR}{CR'}=1.$$

17. P is any point on AB; the median AA′ of the triangle ABC cuts PC at H; prove that $\frac{\text{PH}}{\text{HC}}=\frac{\text{AP}}{\text{AB}}$.

18. P, Q, R are points on the sides BC, CA, AB of a triangle such that $\frac{\text{BP}}{\text{PC}}=\frac{\text{CQ}}{\text{QA}}=\frac{\text{AR}}{\text{RB}}=k$; AP meet QR at D; prove that $\text{RD}=k^2 . \text{DQ}$.

19. Use Theorem 8 to prove Menelaus' Theorem, that if a straight line meets the sides BC, CA, AB of a triangle at L, M, N, then

$$\text{BL} . \text{CM} . \text{AN}=\text{CL} . \text{MA} . \text{NB}.$$

20. Two straight lines ABCD, *abcd* cut four concurrent straight lines OA*a*, OB*b*, OC*c*, OD*d*; prove that

$$\frac{\text{AB}}{\text{BC}}:\frac{\text{AD}}{\text{CD}}=\frac{ab}{bc}:\frac{ad}{cd}.\quad \text{[Use Theorem 8.]}$$

21. P, Q, R are fixed points on the edges BC, CA, AB of a tetrahedron ABCD; a variable plane meets the edges DA, DB, DC at A′, B′, C′ and the lines DP, DQ, DR at P′, Q′, R′; prove that $\frac{\text{B}'\text{P}'}{\text{P}'\text{C}'}\cdot\frac{\text{C}'\text{Q}'}{\text{Q}'\text{A}'}\cdot\frac{\text{A}'\text{R}'}{\text{R}'\text{B}'}$ is constant. Generalise this result for any pyramid.

22. D, E are points on the sides AB, AC of a triangle; Q is the mid-point of DE; AQ meets BC at R; prove that $\frac{\text{BR}}{\text{RC}}=\frac{\text{AB} . \text{AE}}{\text{AC} . \text{AD}}$.

23. Points P, Q, R are taken on BC, CA, AB, so that

$$\frac{\text{BP}}{\text{PC}}=\frac{\text{CQ}}{\text{QA}}=\frac{\text{AR}}{\text{RB}}=\frac{m}{n};$$

prove that the area of the triangle $\text{PQR}=\frac{m^2+n^2}{(m+n)^2}\Delta$, where $\Delta=$ area of triangle ABC.

THEOREM 9.

A, B are two fixed points; P is a moving point such that $\frac{PA}{PB}$ is constant; then the locus of P is a circle

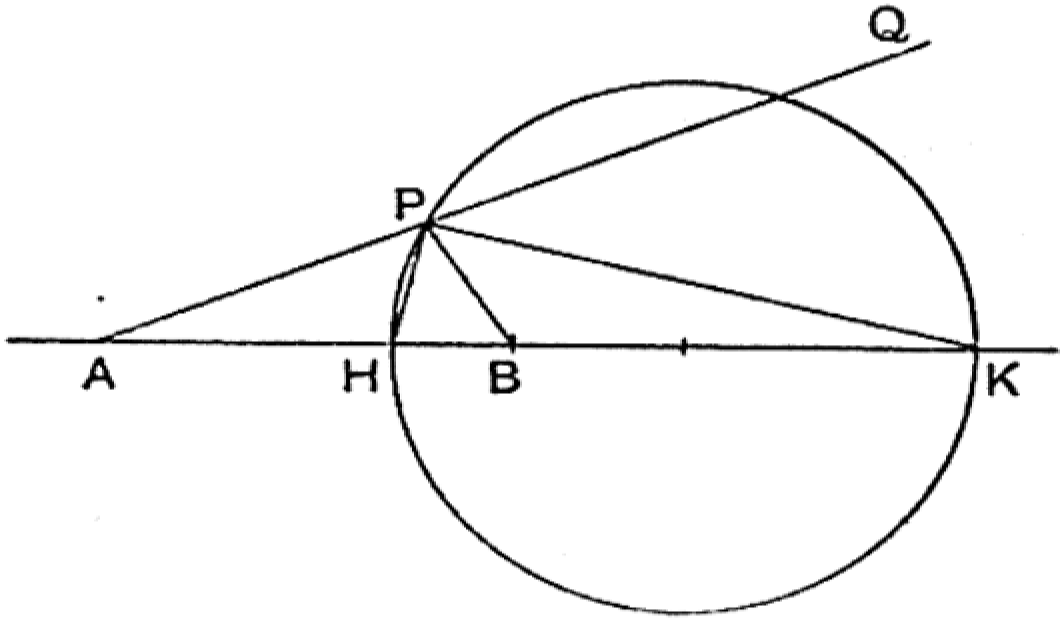

FIG. 14.

Produce AP to Q. Let $\frac{PA}{PB} = \lambda$.

Divide AB internally at H and externally at K in the ratio λ.

Since $\frac{AH}{HB} = \lambda = \frac{AP}{PB} = \frac{AK}{BK}$, PH and PK are the internal and external bisectors of $A\hat{P}B$.

$$\therefore\ H\hat{P}K = \tfrac{1}{2}[A\hat{P}B + B\hat{P}Q] = 90^\circ.$$

$\therefore$ P lies on the circle whose diameter is HK. **Q.E.D.**

This circle is called the circle of *Apollonius.*

For the converse property, see the Corollary of Theorem 35, p. 68.

THEOREM 10. [Ptolemy's Theorem.]

If ABCD is a cyclic quadrilateral, then

$$AB\ .\ CD + BC\ .\ DA = AC\ .\ BD.$$

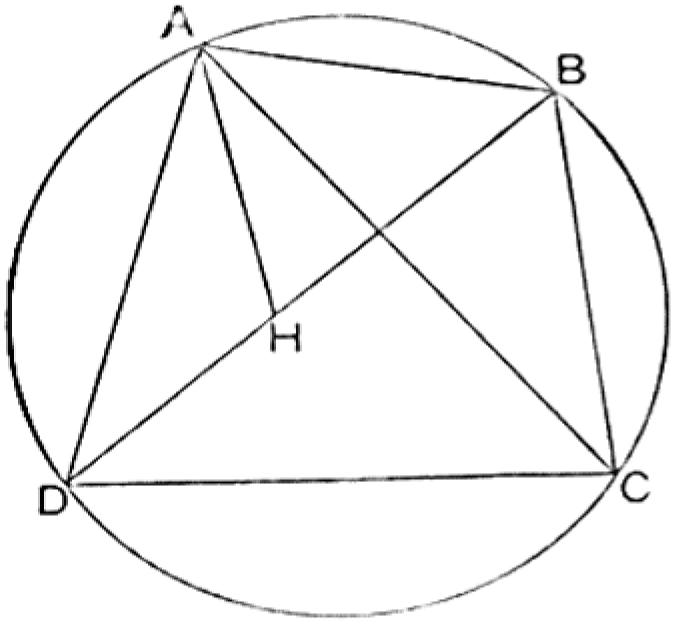

FIG. 15.

Draw AH so that $D\hat{A}H = B\hat{A}C$, and let it meet BD at H.

The triangles ADH, ABC are similar ; $D\hat{A}H = B\hat{A}C$ and $A\hat{D}H = A\hat{C}B$

$$\therefore\ \frac{AD}{DH} = \frac{AC}{CB} \quad \text{or} \quad AD\ .\ CB = AC\ .\ DH.$$

Also the triangles ADC, ABH are similar :

$$D\hat{C}A = H\hat{B}A \quad \text{and} \quad D\hat{A}C = D\hat{A}H + H\hat{A}C = B\hat{A}C + H\hat{A}C = B\hat{A}H.$$

$$\therefore\ \frac{DC}{CA} = \frac{HB}{BA} \quad \text{or} \quad AB\ .\ CD = AC\ .\ HB.$$

$$\therefore\ AB\ .\ CD + AD\ .\ CB = AC\ .\ DH + AC\ .\ HB = AC(DH + HB)$$
$$= AC\ .\ DB.$$

Q.E.D.

24. ABC is a given triangle ; construct a point P such that

$$PA : PB : PC = 1 : 2 : 4.$$

25. ABCD are four collinear points ; construct a point P at which AB, BC, CD subtend equal angles.

26. Given the base, vertical angle and the ratio of the sides containing it, construct the triangle.

27. AB is a chord of a circle perpendicular to a diameter DE and cutting it at H ; the tangent at A meets DE at T ; prove that

$$\frac{TE}{EH} = \frac{TD}{HD}.$$

28. A, B are two fixed points; AP, BQ are parallel chords of a variable circle, such that $\frac{AP}{BQ}$ is constant; prove that the locus of P is a circle.

29. Find the locus of a point at which two given circles subtend equal angles.

30. (i) With the notation of Theorem 9, describe the change in magnitude and position of the circle of Apollonius as λ increases from 0 to 1 and then increases without limit.

(ii) If D is the mid-point of AB and if X is the centre of the circle of Apollonius, prove that (*a*) $XP = \frac{\lambda}{\lambda^2 \sim 1} AB$; (*b*) $DX = \frac{\lambda^2 + 1}{\lambda^2 \sim 1} DB$; (*c*) the length of the tangent from D to the circle of Apollonius is independent of λ.

31. A, B, C are three fixed collinear points; a straight line AP revolves about A, and P is a point on it such that BP, CP are equally inclined to AP; prove that the locus of P is a circle through A.

32. If two figures are similar, find a point O such that by a rigid rotation about O, one of the figures can be made homothetic to the other.

33. The internal and external bisectors of a triangle are drawn and form intercepts on the opposite sides. The three circles are described which have these intercepts as diameters; prove that they have two common points.

34. P is a point on the minor arc BC of the circumcircle of an equilateral triangle ABC; prove that PA = PB + PC.

35. In the triangle ABC, AB = AC; the altitude AD of the triangle meets the circumcircle at P; prove that AP . BC = 2AB . BP.

36. P is a point on the minor arc AB of the circumcircle of the square ABCD; prove that $\frac{PA + PC}{PB + PD} = \frac{PD}{PC}$.

37. P is a point on the minor arc AB of the circumcircle of the regular pentagon ABCDE; prove that $\frac{PA + PD}{PB + PD} = \frac{PE}{PC}$.

38. P is a point on the minor arc AB of the circumcircle of the regular hexagon ABCDEF; prove that PE + PD = PA + PB + PC + PF.

39. If the quadrilateral ABCD is not cyclic, prove that

$$AD . CB + AB . CD > AC . DB.$$

[Draw AH and DH so that $D\hat{A}H = B\hat{A}C$ and $A\hat{D}H = A\hat{C}B$.]

40. P is a point on the minor arc AB of the circumcircle of the regular pentagon ABCDE; prove that PA + PB + PD = PC + PE.

41. P is a point inside a parallelogram ABCD, such that

$$A\hat{P}B + C\hat{P}D = 180° ;$$

prove that
$$AP . CP + BP . DP = AB . BC.$$

CHAPTER III.

PROPERTIES OF THE TRIANGLE.

Notation.

It is convenient to denote particular points connected with the triangle ABC by definite letters.

The following notation will be adopted in this chapter:

a, b, c denote the lengths of BC, CA, AB.

$s = \frac{1}{2}(a+b+c)$; Δ = area of triangle ABC.

A′, B′, C′ are the mid-points of BC, CA, AB.

D, E, F are the feet of the perpendiculars from A, B, C to BC, CA, AB.

I is the in-centre.

r is the radius of the in-circle.

$\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ are the centres of the circles escribed to BC, CA, AB.

r_1, r_2, r_3 are the radii of these circles.

O is the circumcentre.

R is the radius of the circumcircle.

H is the orthocentre.

G is the centroid.

N is the nine-point centre.

X, Y, Z are the points of contact of the in-circle with BC, CA, AB.

X_1, Y_1, Z_1 are the points of contact of the ex-circle, centre $\mathbf{I}_1$, with BC, CA, AB, and similarly for X_2, Y_2, Z_2 and X_3, Y_3, Z_3.

THEOREM 11.

The perpendicular bisectors of the sides of a triangle are concurrent, and their point of intersection is equidistant from the vertices of the triangle.

The proof of this theorem is omitted, as it forms part of every elementary course.

The point of intersection O is called the *circumcentre*, since it is the centre of the circle circumscribing the triangle.

THEOREM 12.

The altitudes of a triangle are concurrent.

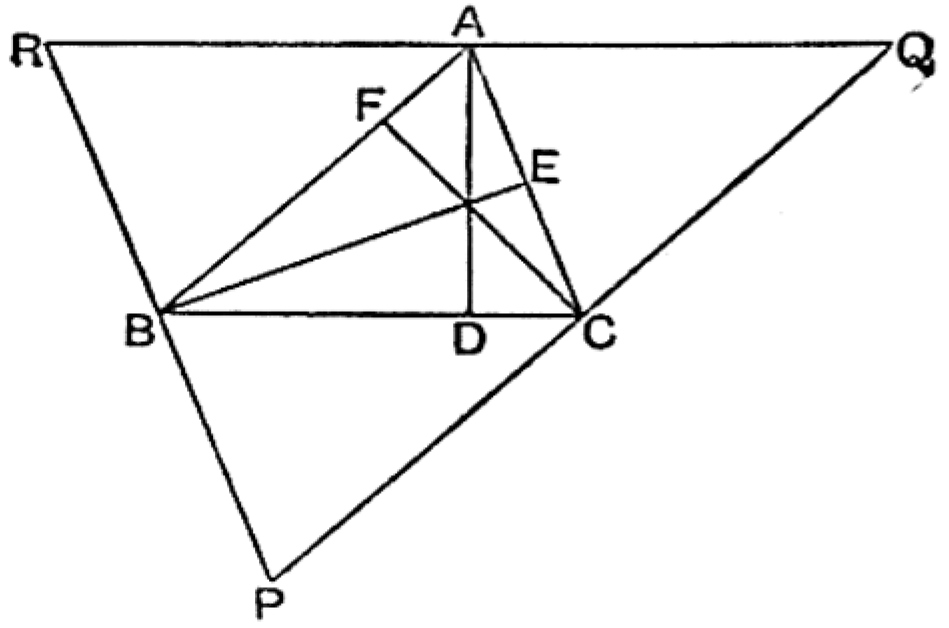

FIG. 16.

Let ABC be the triangle and AD, BE, CF its altitudes.

Through A, B, C draw lines parallel to BC, CA, AB to form the triangle PQR.

Since RBCA is a parallelogram, RA = BC.

Since ABCQ is a parallelogram, AQ = BC.

$$\therefore\ RA = AQ.$$

Since RQ is parallel to BC, AD is perpendicular to RQ.

$\therefore$ AD is the perpendicular bisector of RQ.

Similarly, BE, CF are the perpendicular bisectors of RP, PQ.

$\therefore$ AD, BE, CF are concurrent. [Th. 11.] Q.E.D.

The point H, at which the altitudes concur, is called the *orthocentre* of the triangle: and the triangle DEF is called the *pedal triangle* of the triangle ABC.

The line joining a vertex of a triangle to the mid-point of the opposite side is called a *median* of the triangle.

THEOREM 13.

The three medians of a triangle are concurrent: and the point of intersection is a point of trisection of each median.

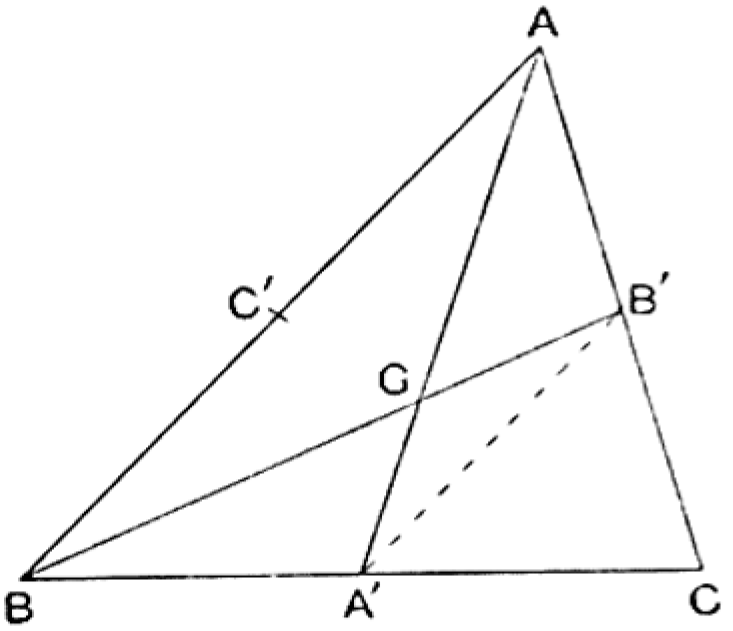

FIG. 17.

Let the medians AA′, BB′ of the triangle ABC intersect at G.

Since $CA' = \frac{1}{2}CB$ and $CB' = \frac{1}{2}CA$,

A′B′ is parallel to AB and $A'B' = \frac{1}{2}AB$.

∴ the triangles A′GB′, AGB are similar.

$$\therefore \frac{A'G}{GA} = \frac{A'B'}{AB} = \frac{1}{2} \quad \text{or} \quad A'G = \frac{1}{2}GA.$$

$$\therefore A'G = \tfrac{1}{3}A'A.$$

∴ BB′ cuts AA′ at the point of trisection of AA′ nearest to A′.

Similarly the median CC′ cuts AA′ at this same point. Q.E.D.

The point at which the medians concur is called the *centroid* of the triangle.

1. Prove that $B\hat{O}A' = B\hat{A}C$.

2. If ABCD is a parallelogram, prove that the circumcentres of the triangles ABC, ADC are equidistant from AC.

3. L is a point in the base BC of an isosceles triangle ABC; P, Q are the circumcentres of the triangles ABL, ACL; prove that PL is parallel to AQ.

4. Find a point P such that the feet of the perpendiculars from P to the sides of a triangle are the angular points of an equilateral triangle.

5. Prove that the perpendicular bisector of EF bisects BC.

6. If A, B, C are three collinear points and K any other point, prove that the circumcentres of the triangles KAB, KBC, KCA are concyclic with K

7. The diagonals AC, BD of a quadrilateral ABCD meet at K; prove that the circumcentres of the triangles KAB, KBC, KCD, KDA are the vertices of a parallelogram.

8. P, Q, R are any points on the sides BC, CA, AB of a triangle; prove that the circumcircles of the triangles AQR, BRP, CPQ are concurrent.

9. The extremities of a straight line BC of constant length slide on two fixed lines AB, AC; find the locus of the circumcentre of the triangle ABC.

10. Two equivalent triangles ABC, XYZ are inscribed in the same circle; prove that $AB \cdot BC \cdot CA = XY \cdot YZ \cdot ZX$.

11. Prove that $A\hat{H}E = A\hat{C}B$.

12. Prove that the triangles AEF, ABC are similar.

13. Prove that the triangles BDF, EDC are similar.

14. Prove that $AH \cdot HD = BH \cdot HE$.

15. Prove that $AF \cdot AB = AH \cdot AD$.

16. Prove that $F\hat{D}A = E\hat{D}A$.

17. Prove that $F\hat{D}E = 180^\circ - 2B\hat{A}C$.

18. Prove that $B\hat{H}C + B\hat{A}C = 180^\circ$ or $B\hat{H}C = B\hat{A}C$.

19. If S is the orthocentre of the triangle PQR, prove that P is the orthocentre of the triangle QRS.

20. Given the base BC and the angle BAC of the variable triangle ABC, find the locus of the orthocentre, and construct its position when A is very close to B.

21. DP, DQ are the perpendiculars from D to AC, AB; prove that C, P Q, B are concyclic and that $B\hat{P}D = C\hat{Q}D$.

22. Prove that the circumcentre of the triangle ABC is the orthocentre of the triangle A′B′C′.

23. Three equal circles have a common point: prove that it is the orthocentre of the triangle formed by the other intersections.

24. Prove that OB is perpendicular to FD.

25. P is a point on the base BC of an isosceles triangle ABC; K, L are the orthocentres of the triangles APB, APC. Prove that PKL is an isosceles triangle.

26. If $FB = FE$, prove that $AE = BD$.

27. Through the vertices of a triangle are drawn six equal lines terminated by the opposite sides; prove that their mid-points are all equidistant from the orthocentre.

28. If AG is produced to K so that $AG = GK$, prove that $BK = CG$.

29. Construct a triangle, given (i) the lengths of the three medians; (ii) the lengths of one side and two medians.

30. If $AG = BC$, prove that $B\hat{G}C = 90^\circ$.

31. If B′C′ meets AA′ at P, calculate the ratio $\frac{PG}{PA'}$.

32. Prove that the area of the triangle whose sides are equal to AA′, BB′, CC′ equals $\frac{3}{4}\Delta$.

33. In a tetrahedron ABCD, the plane angles at each of three corners add up to 180°, prove that the opposite edges are equal. [Cut down the edges AB, AC, AD and fold it out flat. *Mathematical Gazette.*]

34. If K is the mid-point of EF, prove that $K\hat{A}C = A'\hat{A}B$.

35. ALMB, APQC are the squares described externally on the sides AB, AC of the triangle ABC; prove that the median KA of the triangle LAP is perpendicular to BC.

THEOREM 14.

The internal bisectors of a triangle are concurrent; and each is concurrent with the external bisectors of the other two angles. The four points so obtained are each equidistant from the sides of the triangle and are the centres of the inscribed circle and the three escribed circles of the triangle.

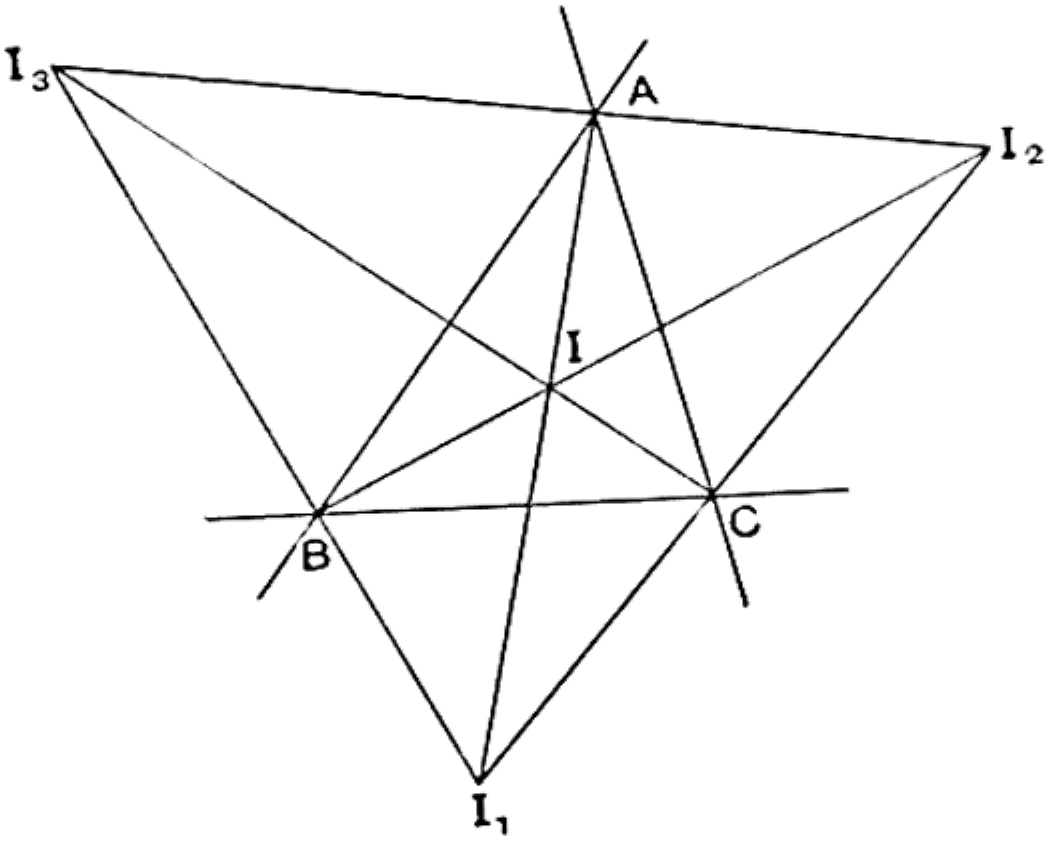

FIG. 18.

The proof of this theorem is omitted as it forms part of every elementary course.

The four points **I**, $\mathbf{I_1}$, $\mathbf{I_2}$, $\mathbf{I_3}$ are called the *in-centre* and *ex-centres* of the triangle.

Notice that **I** is the orthocentre of the triangle $\mathbf{I_1I_2I_3}$.

THEOREM 15.

The in-circle of the triangle ABC touches BC, CA, AB at X, Y, Z and the circle escribed to BC touches BC, CA, AB at X_1, Y_1, Z_1, then

(i) $AZ = AY = s - a$;

(ii) $AZ_1 = AY_1 = s$;

(iii) $BX_1 = CX$.

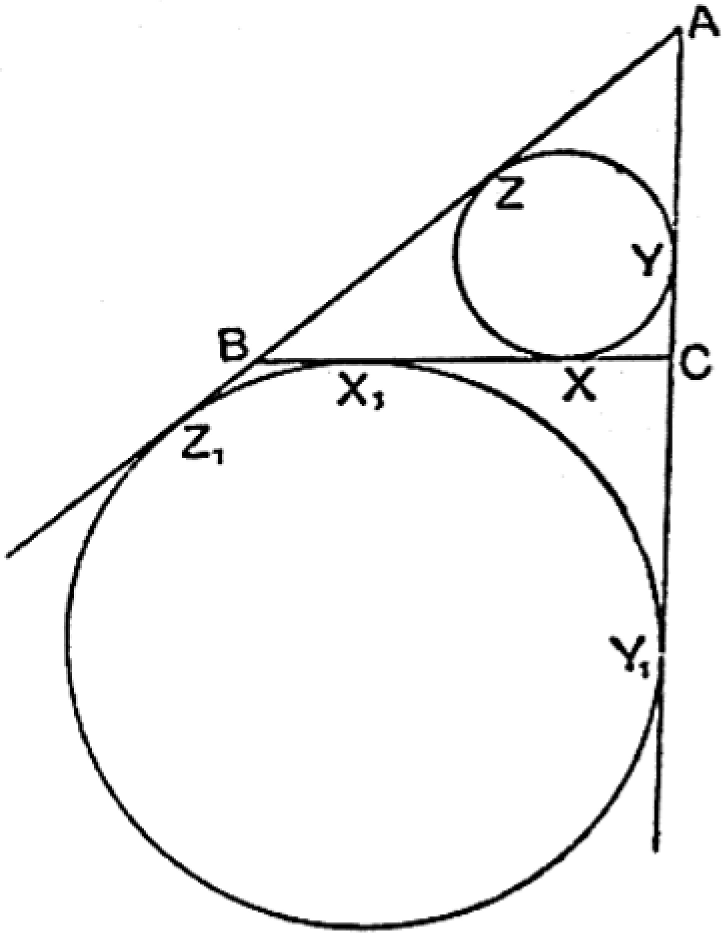

FIG. 19.

(i) $2s = a + b + c = a + (AY + YC) + (AZ + ZB)$

$= a + AY + XC + AY + BX$

$= a + 2AY + BC = 2a + 2AY.$

$\therefore\ AY = s - a.$

(ii) $2AZ_1 = AZ_1 + AY_1 = AB + BZ_1 + AC + CY_1$

$= AB + BX_1 + AC + X_1C = AB + AC + BC = 2s.$

$\therefore\ AZ_1 = s.$

(iii) $BX_1 = BZ_1 = AZ_1 - AB = s - c = XC.$ Q.E.D.

THEOREM 16.

With the usual notation, $r = \frac{\Delta}{s}$ and $r_1 = \frac{\Delta}{s-a}$.

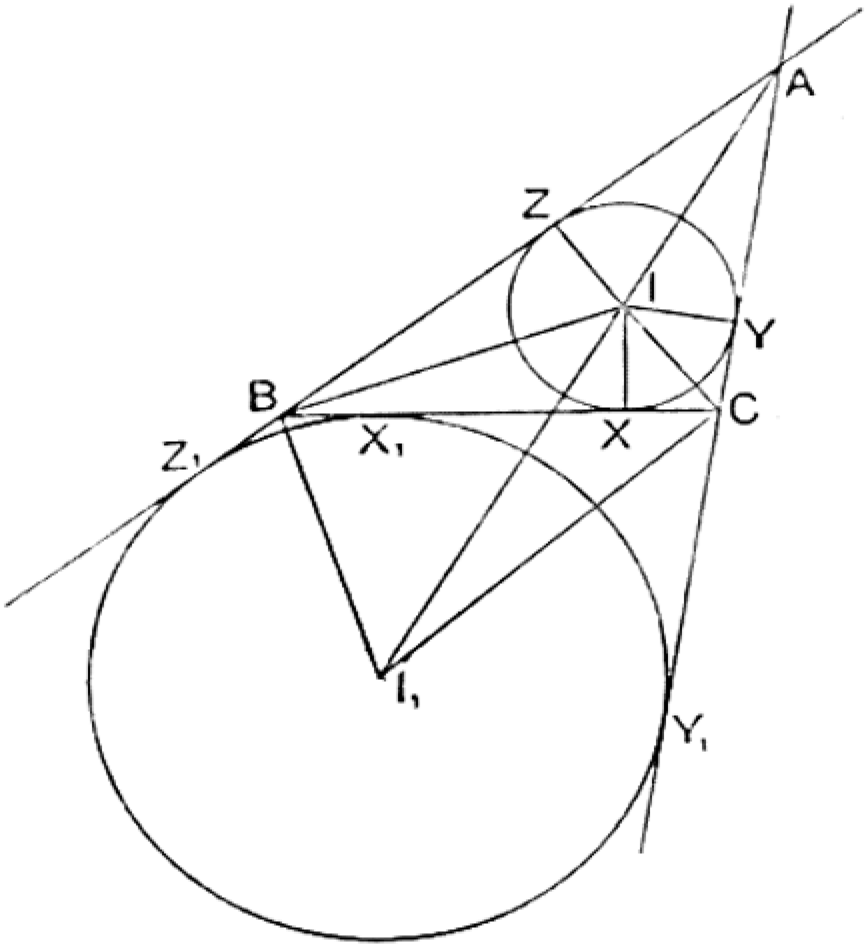

FIG. 20.

(i) Δ = triangle IBC + triangle ICA + triangle IAB

$= \frac{1}{2}IX \cdot BC + \frac{1}{2}IY \cdot CA + \frac{1}{2}IZ \cdot AB$

$= \frac{1}{2}r \cdot a + \frac{1}{2}r \cdot b + \frac{1}{2}r \cdot c = \frac{1}{2}r(a+b+c)$

$= \frac{1}{2}r(2s) = rs.$

$$\therefore\ r = \frac{\Delta}{s}.$$

(ii) Δ = triangle I_1AB + triangle I_1AC − triangle I_1BC

$= \frac{1}{2}I_1Z_1 \cdot AB + \frac{1}{2}I_1Y_1 \cdot AC - \frac{1}{2}I_1X_1 \cdot BC$

$= \frac{1}{2}r_1(c+b-a) = \frac{1}{2}r_1(c+b+a-2a) = \frac{1}{2}r_1(2s-2a)$

$= r_1(s-a).$

$$\therefore\ r_1 = \frac{\Delta}{s-a}.$$

Q.E.D

36. Prove that (i) $YY_1 = ZZ_1 = a$;

(ii) $XX_1 = c \sim b$;

(iii) $Z_1Z_2 = a + b$.

37. From any point inside a regular polygon, perpendiculars are drawn to the sides of the polygon, prove that the sum of their lengths is constant.

38. Prove that the area of the quadrilateral I_1YAZ is equal to Δ.

39. If $B\hat{A}C = 60°$, prove that O, H, I, I_1, B, C lie on a circle.

40. Prove that $I\hat{A}O$ is half the difference of the angles ABC, ACB.

41. M is the foot of the perpendicular from C to AI; prove that the angles BCM, ACM are respectively equal to half the difference and sum of the angles ACB, ABC.

42. Prove that $B\hat{I}C = 90° + \frac{1}{2}A$.

43. Prove that the circum-radius of the triangle $AYZ = \frac{1}{2}AI$.

44. P, Q are the feet of the perpendiculars from X, Y to YZ, ZX, prove that PQ is parallel to AB.

45. Given the base and vertical angle of a triangle, find the locus of the in-centre.

46. If $A\hat{C}B = 90°$, prove that $2r = a + b - c$ and that $r_3 = s$.

47. The in-circle of the triangle ABC touches the sides at A_1, B_1, C_1; the in-circle of $A_1B_1C_1$ touches the sides at A_2, B_2, C_2 and so on; prove that $B_n\hat{A}_nC_n = 60° + (-2)^{-n}[B\hat{A}C - 60°]$.

48. If P, Q, R are the circumcentres of the triangles BIC, CIA, IAB, prove that $Q\hat{P}R = 90° - \frac{1}{2}A$.

49. A variable line XY cuts two fixed lines AB, CD at X, Y; the lines bisecting the angles AXY, CYX meet at P; find the locus of P.

50. Prove that H is the in-centre of the pedal triangle of ABC. Where are its ex-centres?

51. K, L are the feet of the perpendiculars from A to BI, CI; prove that KL is parallel to BC.

52. From a given point, draw a line so as to form with two given lines a triangle of given perimeter.

53. K is the circumcentre of the triangle $I_1I_2I_3$, prove that KI_1 is perpendicular to BC.

54. Two sides of a triangle of given perimeter are given in position, find the envelope of the third side.

55. If $A\hat{B}C = 90°$, prove that $AY \cdot YC = \Delta$.

56. If the circle BIC cuts AB, AC again at M, N, prove that MN touches the in-circle.

57. AD is an altitude of the triangle ABC, right-angled at A. x, y, z are the in-radii of the triangles ADC, ADB, ABC; prove that $x^2 + y^2 = z^2$.

58. ABCD is a quadrilateral circumscribing a circle, prove that the in-circles of the triangles ABC, ADC touch each other.

THEOREM 17.

With the usual notation, if P, Q, R are the mid-points of HA, HB, HC, then the nine points A′, B′, C′, D, E, F, P, Q, R lie on a circle whose radius equals ½R.

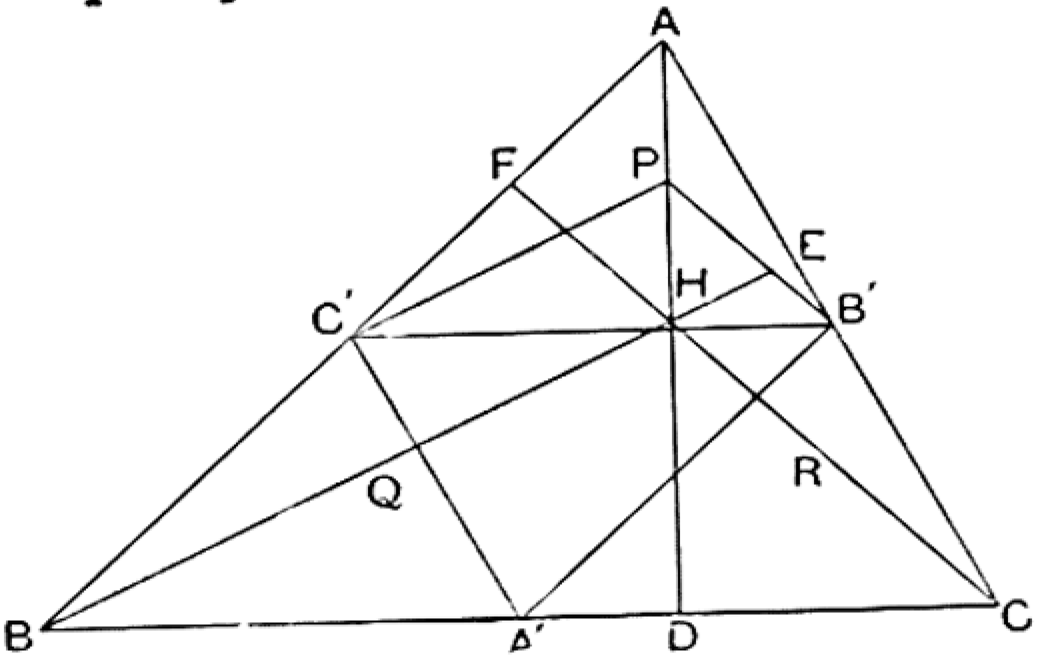

FIG. 21.

Since AP = PH and AC′ = C′B, ∴ C′P is parallel to BH.
Since BC′ = C′A and BA′ = A′C, ∴ A′C′ is parallel to AC.
But BH is perpendicular to AC, ∴ C′P is perpendicular to A′C′.

$$\therefore\ A'\hat{C}'P = 90^\circ.$$

Similarly $A'\hat{B}'P = 90^\circ$; but $A'\hat{D}P = 90^\circ$.
∴ the circle on A′P as diameter passes through B′, C′, D.

∴ D and P lie on the circle A′B′C′.

Similarly E, Q and F, R lie on the circle A′B′C′.
∴ the nine points A′, B′, C′, D, E, F, P, Q, R lie on a circle.
Moreover each side of the triangle A′B′C′ is half the corresponding side of the triangle ABC and ∴ the radius of the circle A′B′C equals ½R. Q.E.D.

The circle which passes through these nine points is called the *nine-point circle* and its centre N is called the *nine-point centre.*

THEOREM 18.

The circumcentre, centroid, orthocentre and nine-point centre of a triangle lie on a straight line (called the Euler line of the triangle).

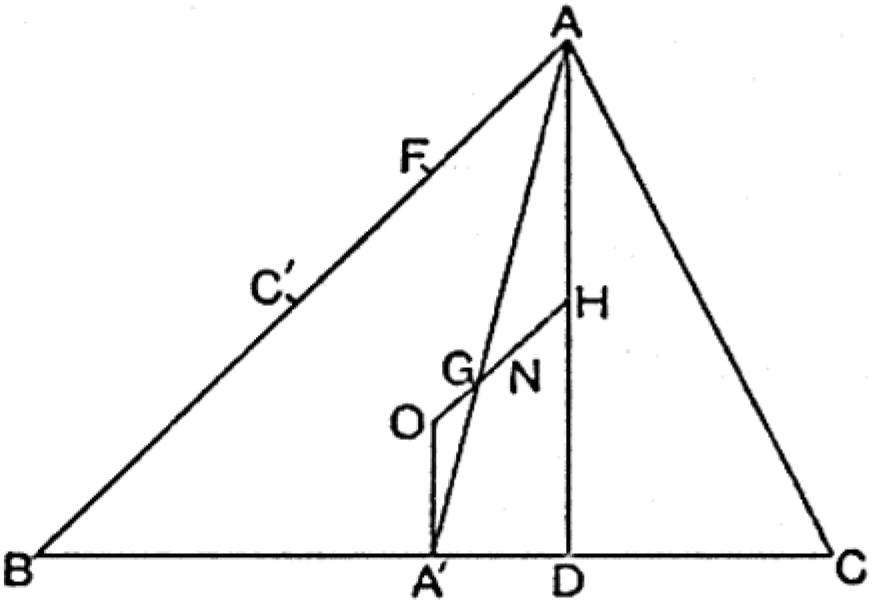

FIG. 22.

Let O, G be circumcentre and centroid.

Produce OG to H so that $OG = \frac{1}{2}GH$. We shall prove that H is the orthocentre.

Join AH and produce it to meet BC at D.

Since $A'G = \frac{1}{2}GA$, $\frac{A'G}{GA} = \frac{OG}{GH}$.

∴ AH is parallel to OA′.

But OA′ is perpendicular to BC.

∴ AH is perpendicular to BC.

Similarly CH is perpendicular to AB.

∴ H is the orthocentre.

Again, since A′O and DH are perpendicular to BC, the perpendicular bisector of A′D bisects OH.

Similarly, the perpendicular bisector of C′F bisects OH,

But A′D and C′F are chords of the nine-point circle,

∴ the mid-point of OH is the centre of the nine-point circle.

Q.E.D.

The following deductions should be noted :

(1). Since $OG = \frac{1}{2}GH$ and $ON = NH$, *it follows that*

$$OG : GN : NH = 2 : 1 : 3.$$

(2). From the similar triangles OGA', HGA, $\frac{OA'}{AH} = \frac{A'G}{GA} = \frac{1}{2}$.

Therefore $AH = 2OA'$.

(3). Since $\frac{OG}{GN} = \frac{OH}{NH} = \frac{2}{1}$, G *and* H *are the centres of similitude of the circumcircle and nine-point circle.*

59. If P is the mid-point of AH, prove that $A'\hat{P}C' = A'B'C'$.

60. Prove that the radius of the circumcircle of the pedal triangle equals $\frac{1}{2}R$.

61. Prove that the triangles ABC, HBC have the same nine-point circle.

62. If P is the mid-point of AH, prove that $F\hat{P}C'$ equals the difference of the angles BAC, ABC.

63. Prove that I_1I_2 is bisected by the circum-circle of the triangle ABC.

64. If P, Q, R are the mid-points of HA, HB, HC, prove that $A'P$, $B'Q$, $C'R$ are concurrent.

65. Prove that the tangent to the nine-point circle at the mid-point of the base is parallel to the tangent to the in-circle from the point at which the bisector of the vertical angle meets the base.

66. Prove that the nine-point circle of the triangle $AB'C'$ touches the nine-point circle of the triangle $A'B'C'$.

67. Given the base and vertical angle of a triangle ABC, prove that the locus of N is a circle, centre A'.

68. Prove that the radius of the circle II_1I_2 is equal to 2R.

69. Prove that the perimeter of the pedal triangle is equal to $\frac{2\Delta}{R}$;

70. Prove that the area of the triangle $I_1I_2I_3$ is equal to $2Rs$.

71. Prove that if OA', OB', OC' are placed end to end on the circumference of the nine-point circle, they will just fill half of it.

72. If P is the mid-point of AH, prove that $A'\hat{P}D = A\hat{C}B \sim A\hat{B}C$.

73. If L is the mid-point of EF, prove that $A'L$ passes through N.

74. If $B\hat{A}C = 60°$, prove that the Euler line forms with AB, AC an equilateral triangle.

75. Prove that $AH^2 + BC^2 = 4AO^2$.

76. If $BAC = 45°$, prove that EF bisects OH.

77. If P is the mid-point of AH, prove that $A'P$ is perpendicular to EF.

78. Prove that NA′ is parallel to OA.

79. If DG is produced to meet the circumcircle at Q, prove that OQ is parallel to ND.

80. The extremities of a straight line PQ of constant length move on two fixed lines AB, AC; prove that the locus of the orthocentre of APQ is a circle, centre A.

81. Given A, N H construct the triangle ABC.

82. Prove that HA′ meets the circumcircle at the end P of the diameter through A of the circumcircle.

83. If DG is produced to meet the circumcircle at Q, prove that $Q\hat{A}D = 90°$.

84. If P is the mid-point of AH and if PG meets the circumcircle at Q, prove that PA′ is parallel to AQ.

85. If P, Q, R are the reflections of the circumcentre in the sides BC, CA, AB of the triangle ABC, prove that AP, BQ, CR are concurrent.

THEOREM 19.

In the triangle ABC, AI bisects the angle OAH.

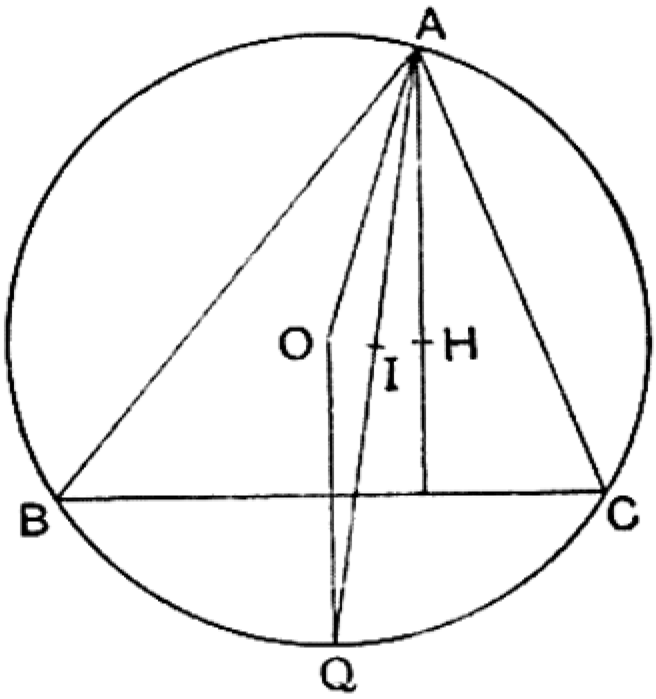

FIG. 23.

Let AI meet the circumcircle in Q.

Since $B\hat{A}Q = C\hat{A}Q$, arc BQ = arc QC.

∴ OQ is perpendicular to BC and ∴ parallel to AH.

∴ $H\hat{A}I = I\hat{Q}O$, alternate angles

$= O\hat{A}I$, since OQ = OA radii. Q.E.D.

THEOREM 20.

If the altitude AD meets the circumcircle of the triangle ABC at P, and if H is the orthocentre, then HD = DP.

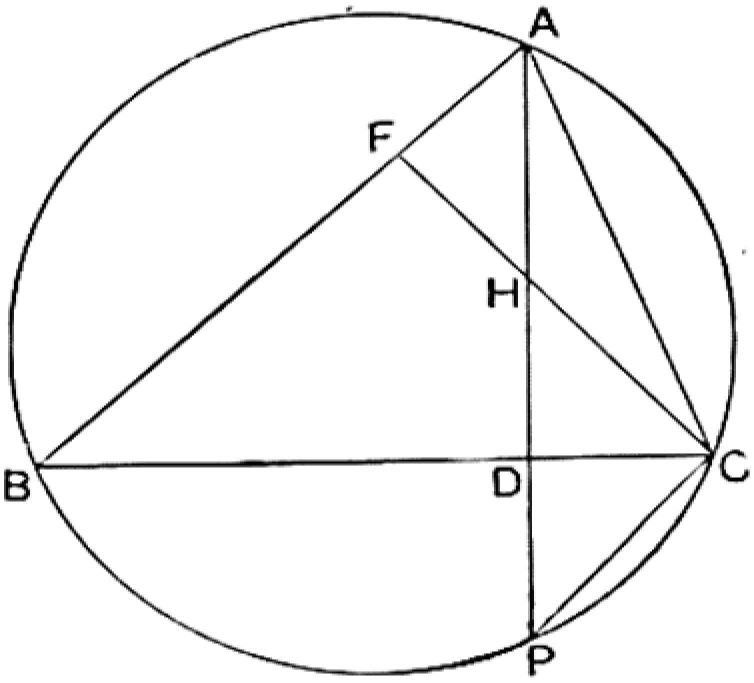

FIG. 24.

Join CP and draw the altitude CHF.

$$P\hat{C}D = P\hat{A}B \text{ in the same segment}$$
$$= 90^\circ - D\hat{B}A, \quad \text{since } A\hat{D}B = 90^\circ$$
$$= B\hat{C}F, \quad \text{since } B\hat{F}C = 90^\circ.$$

∴ the triangles HDC, PDC are congruent, for $H\hat{C}D = P\hat{C}D$ just proved and $H\hat{D}C = 90^\circ = P\hat{D}C$.

∴ HD = DP. Q.E.D.

THEOREM 21.

H is the in-centre of the pedal triangle DEF of the triangle ABC.

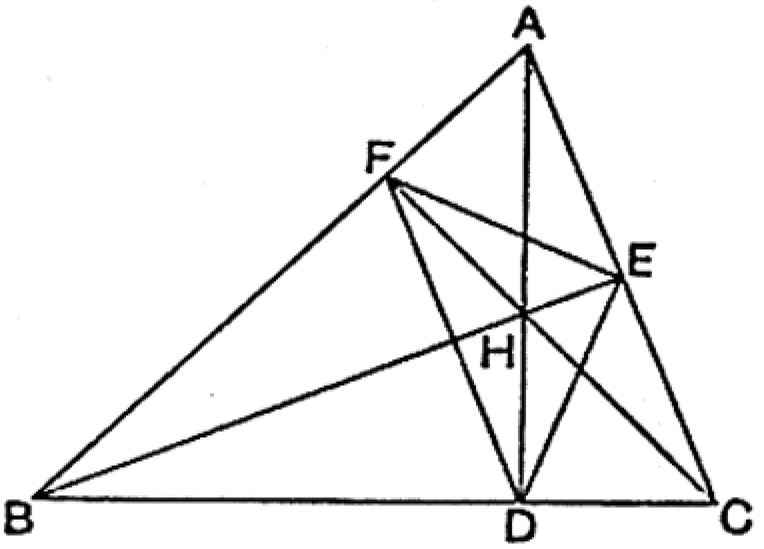

Fig. 25.

$\hat{BFH} = 90° = \hat{BDH}$; $\therefore$ BFHD is a cyclic quadrilateral.

$$\therefore \hat{FDH} = \hat{FBH} = \hat{ABE}$$

$$= 90° - \hat{BAC}.$$

Similarly $\hat{EDH} = ECH = \hat{ACF} = 90° - \hat{BAC}.$

$$\therefore \hat{FDH} = \hat{EDH}.$$

$$\therefore \text{HD bisects } \hat{FDE}.$$

Similarly HE bisects $\hat{DEF}$.

$\therefore$ H is the in-centre of the triangle DEF. Q.E.D.

Notice that, since the triangles AEF, ABC are similar,

$$\frac{EF}{BC} = \frac{AF}{AC} = \cos A; \quad \therefore EF = a \cos A.$$

86. If BE, CF are altitudes of the triangle ABC, prove that the triangles AEF, ABC are similar.

87. If QR is the diameter of the circumcircle of the triangle ABC perpendicular to BC, prove that AQ, AR are the two bisectors of the angle BAC.

88. If AQ is a diameter of the circumcircle of the triangle ABC, prove that HQ bisects BC.

89. Prove that AD . HD = BD . DC.

90. Prove that the circumcircles of the triangles ABC, HBC are equal.

91. AD, BE, CF meet the circumcircle of ABC at P, Q, R; prove that H is the in-centre of PQR.

92. BE, CF meet the circumcircle of the triangle ABC at Q, R: prove that A is the centre of the circle HQR.

93. Prove that the circumcircles of the triangles HCA, HAB are equal.

94. P, Q, R are the circumcentres of the triangles HBC, HCA, HAB; prove that the triangles PQR, ABC are congruent and that H is the circumcentre of PQR.

95. ABCD is a cyclic quadrilateral; H, K are the orthocentres of the triangles ABC, ABD; prove that A, B, H, K are concyclic.

96. If BE, CF meet the circumcircle of ABC at Q, R, prove that A is the mid-point of the arc QR.

97. BC is a fixed chord of a circle and P a variable point on the circumference; BE, CF are altitudes of the triangle BPC; prove that EF is of constant length.

98. Prove that A is an ex-centre of the triangle DEF.

99. Prove that $FD \cdot DE = BD \cdot DC$.

100. If CO meets the circumcircle at P, prove that PBHA is a parallelogram and deduce that $AH = 2OA'$.

101. Prove that $O\hat{A}I = \frac{1}{2}(C \sim B)$.

102. Prove that the radius of the in-circle of the triangle BDF $= r \cos B$

THEOREM 22.

If AII_1 meets the circumcircle of the triangle ABC at P, then

$$PI = PB = PC = PI_1.$$

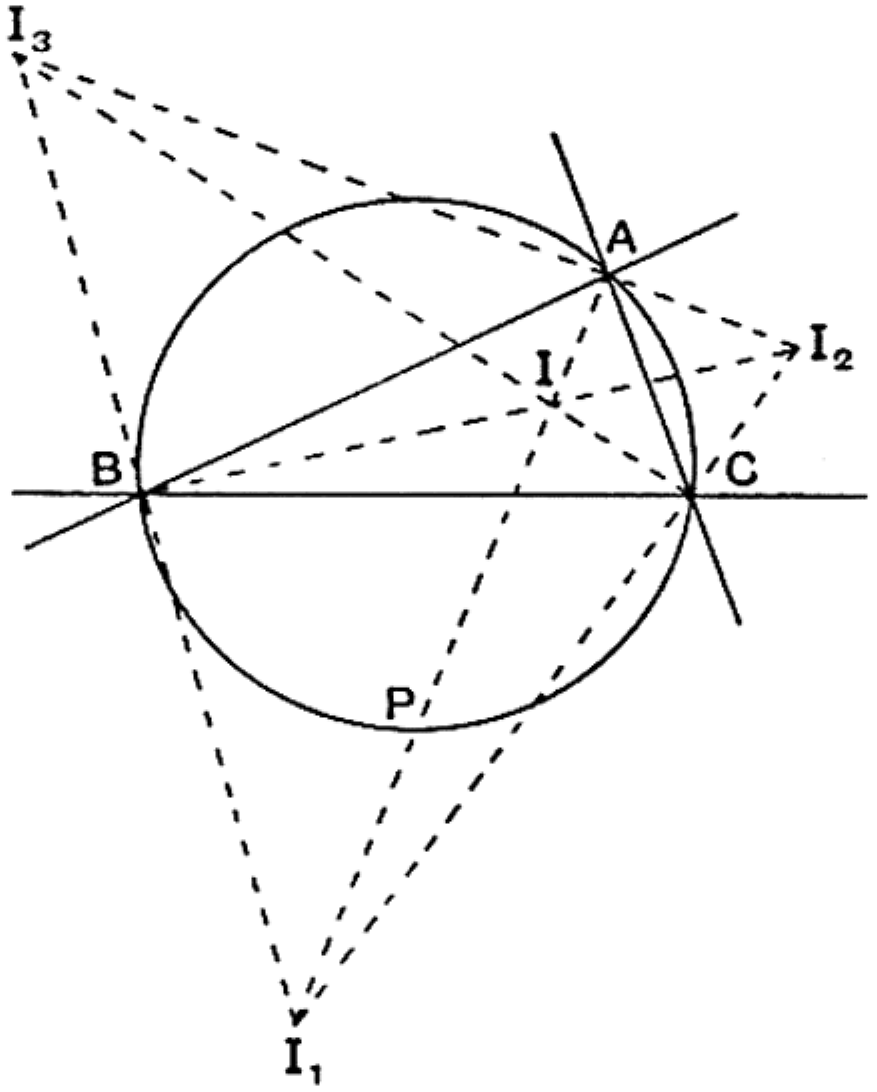

FIG. 26.

I is the orthocentre of the triangle $I_1I_2I_3$ and ABC is the pedal triangle of $I_1I_2I_3$.

$\therefore$ the circumcircle of ABC is the nine-point circle of $I_1I_2I_3$.

$\therefore$ P is the mid-point of II_1.

But since $I\hat{B}I_1 = 90° = I\hat{C}I_1$, the circle on II_1 as diameter (whose centre is P) passes through B, C.

$\therefore$ $PI = PI_1 = PB = PC$. Q.E.D.

THEOREM 23.

With the usual notation, $OI^2 = R^2 - 2Rr$.

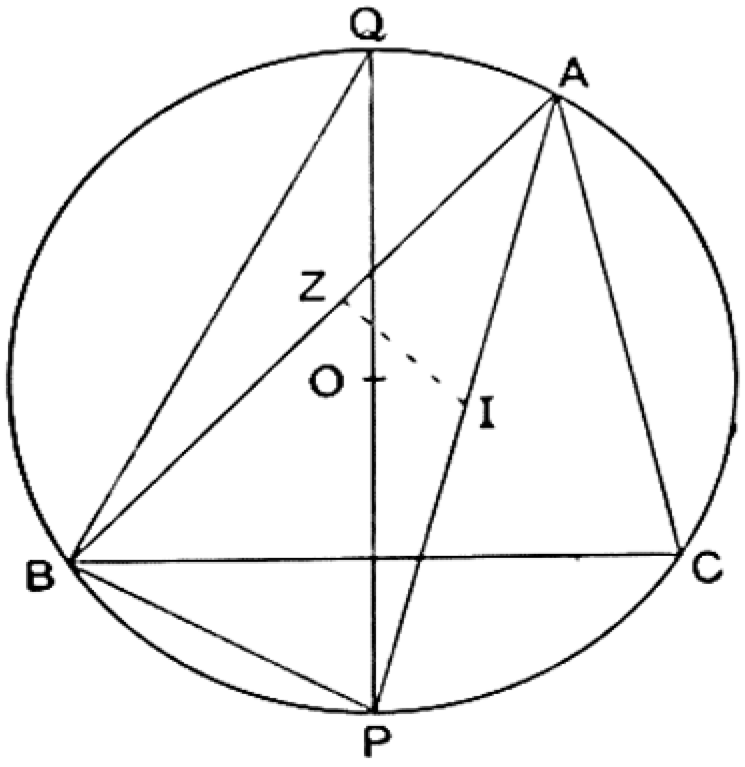

FIG. 27.

Produce AI to meet the circumcircle at P; produce PO to meet the circumcircle at Q; draw IZ perpendicular to AB.

Then $R^2 - OI^2 = AI \, . \, IP$ [Th. 7.]

$= AI \, . \, PB.$ [Th. 22.]

But $AI = IZ \operatorname{cosec} \frac{A}{2} = r \operatorname{cosec} \frac{A}{2}$,

and $PB = PQ \sin \frac{A}{2} = 2R \sin \frac{A}{2}$,

since $B\hat{Q}P = B\hat{A}P = \frac{A}{2}$.

$$\therefore \; R^2 - OI^2 = r \operatorname{cosec} \frac{A}{2} \, . \, 2R \sin \frac{A}{2} = 2R \, . \, r.$$

$$\therefore \; OI^2 = R^2 - 2R \, . \, r.$$ Q.E.D.

By the same method it can be proved that

$$OI_1^2 = R^2 + 2R \, . \, r_1.$$

$[OI_1^2 - R^2 = OI_1^2 - OP^2 = I_1P \, . \, I_1A = PB \, . \, I_1A = 2R \, . \, r_1.]$

103. If the internal bisector of the angle BAC meets the circumcircle of the triangle ABC at P, and if M, N are the feet of the perpendiculars from P to AB, AC, prove that

$$AM = AN = \tfrac{1}{2}(AB + AC) \quad \text{and} \quad BM = CN = \tfrac{1}{2}(AB \sim AC).$$

104. In the triangle ABC, $B\hat{A}C$ is fixed in magnitude and position and $AB + AC$ is constant, prove that the locus of the mid-point of BC is a straight line.

105. R is the mid-point of the minor arc BC of the circumcircle of ABC; the circle on AR as diameter meets AB at S, prove that the tangent from S to the in-circle equals $\tfrac{1}{2}BC$.

106. If AI_1 cuts the circle ABC at P, prove that $P\hat{C}I_1 = \frac{B}{2}$.

107. AI, BI, CI are produced to meet the circle ABC at P, Q, R; prove that I is the orthocentre of PQR.

108. If the circle BIC cuts AC at P, prove that $AP = AB$.

109. Prove that the circumcircle of the triangle formed by any three of the four common tangents to two circles passes through the mid-point of the line joining their centres.

110. I_1O is produced to K so that $I_1O = OK$; prove that $IK = 2OA$.

111. A fixed circle X passes through the centre of another fixed circle Y. A variable triangle circumscribes Y and has two of its vertices on X, prove that the locus of the third vertex is a straight line.

112. A variable triangle circumscribes a fixed circle; if its circumradius is constant, find the locus of its circumcentre.

113. Prove that the ratio of the squares of the tangents from I_1 and I_2 to the circumcircle equals $\frac{r_1}{r_2}$.

114. The perpendicular at I to IO meets the in-circle at K, K′; prove that the perimeter of the triangle OKK′ equals the diameter of the circumcircle.

115. A circle is drawn to touch OI at I and to touch the circumcircle; prove that it is equal to the in-circle.

116. If the circumcircle meets an ex-circle at P, and if the tangents at P to the two circles are at right angles, prove that the radius of the ex-circle equals the diameter of the circumcircle.

CHAPTER IV.

CONCURRENCY AND COLLINEARITY.

Definition.

Two segments AB, PQ of the same or parallel lines are said to have the same *sense* or opposite senses [or are sometimes called *like* or *unlike*] according as the displacements $A \rightarrow B$ and $P \rightarrow Q$ are in the same or opposite directions.

In Coordinate Geometry it is essential to take account of the sense of a line; if lengths measured in one sense along a line are taken as positive, then lengths measured in the opposite sense along that line are taken as negative.

Many geometrical theorems can be enunciated in a more general form, if this conventional connection between sign and sense is observed.

Thus $\qquad AB = -BA \quad$ or $\quad AB + BA = 0.$

This is equivalent to stating that the addition of a displacement from A to B and a displacement from B to A is zero.

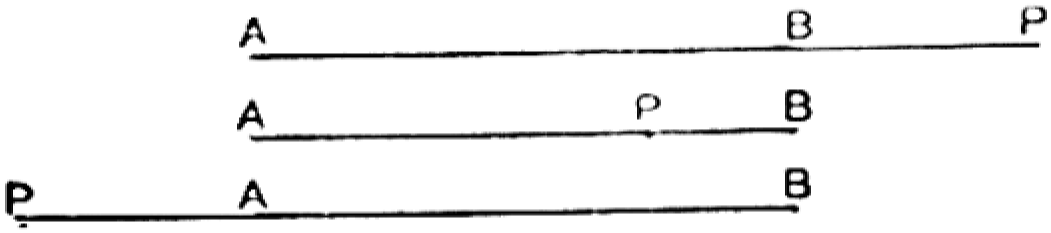

FIG. 28.

If A, B, P are three points *in any order* on a straight line, then $AB = AP + PB$.

If A, B, P_1, P_2, ... P_n are points in any order on a straight line, then

(i) $AB = AP_1 + P_1P_2 + \ldots + P_nB$; (ii) $AP_1 + P_1P_2 + \ldots P_nB + BA = 0.$

If A, B, O are points in any order on a straight line, then

$$AB = OB - OA.$$

1. Draw any finite straight line AB and produce it each way; mark roughly on it positions of P, such that $\frac{AP}{PB}$ equal $1, -\frac{1}{2}, -2, -1$. Describe the changes in the value of the ratio $\frac{AP}{PB}$ as P moves on the unlimited line, and prove that this ratio can never have the same value for two different positions of P.

It appears from Ex. 1, that if x is any real number, positive or negative, excluding $x = -1$, there is one and only one position of P on the line AB for which $\frac{AP}{PB} = x$. And as the distance of P from A or B increases, the ratio $\frac{AP}{PB}$ approaches the value -1, and can be made to differ from -1 by a quantity less than any assigned quantity, however small, by increasing sufficiently the distance of P from A.

If $x = -1$, there is no possible *finite* position for P. To preserve continuity, we shall therefore suppose there is *one and only one* position of P on the line AB for which $\frac{AP}{PB} = -1$ and we shall call this point, *the point at infinity* on the line AB and shall denote it by the symbol ∞.

Such a point is an ideal point, it is really only a convenient way of expressing in geometrical language an analytical idea. Further discussion of this subject will be found in Chapter I., *Plane Geometry*, Part II.

Various relations exist connecting the lengths of segments of a line; one of the most important is the following:

If A, B, C, D are any four collinear points, then

$$AB \,.\, CD + BC \,.\, AD + CA \,.\, BD = 0.$$

Such relations may easily be proved by taking an origin O on the line and expressing the length of each segment in terms of the distances of its extremities from O.

Thus $\quad AB = OB - OA, \quad BC = OC - OB$, etc.

2. C is the mid-point of AB, P is any other point on the line AB, prove that

$$(1)\ \mathrm{PC}=\tfrac{1}{2}(\mathrm{PA}+\mathrm{PB})\,;\quad (2)\ \mathrm{AP}\,.\,\mathrm{BP}+\mathrm{CB}^2=\mathrm{CP}^2\,;$$
$$(3)\ \mathrm{AP}^2+\mathrm{BP}^2=2\mathrm{AC}^2+2\mathrm{CP}^2.$$

3. AB is divided at C so that $\mathrm{BA}\,.\,\mathrm{BC}=\mathrm{AC}^2$; prove that

$$\mathrm{AB}^2-\mathrm{AC}^2=\mathrm{AB}\,.\,\mathrm{AC}.$$

4. AB is bisected at C and produced to D; prove that

$$\mathrm{AC}\,.\,\mathrm{AD}=\mathrm{CB}\,.\,\mathrm{BD}+2\mathrm{BC}^2.$$

5. A, B, C, D are collinear; P, Q are the mid-points of AB, CD; prove that $\mathrm{PQ}=\tfrac{1}{2}(\mathrm{AD}+\mathrm{BC})$.

6. P is any point on the line ABC; prove that

$$\mathrm{PA}^2\,.\,\mathrm{BC}+\mathrm{PB}^2\,.\,\mathrm{CA}+\mathrm{PC}^2\,.\,\mathrm{AB}+\mathrm{BC}\,.\,\mathrm{CA}\,.\,\mathrm{AB}=0.$$

7. Prove the result of Ex. 6 when P does not lie on the line ABC.

8. G is a point on the line $\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\mathrm{A}_4$ such that

$$\mathrm{GA}_1+\mathrm{GA}_2+\mathrm{GA}_3+\mathrm{GA}_4=0\,;$$

if P is any other point on the line, prove that $\Sigma\mathrm{PA}=4\mathrm{PG}$.

9. With the notation and condition of Ex. 8, prove that

$$\Sigma\mathrm{PA}^2-\Sigma\mathrm{GA}^2=4\mathrm{PG}^2.$$

Generalise this for *n* collinear points, and prove that the result still holds when P does not lie on the straight line.

THEOREM 24. [Ceva's Theorem.]

Three concurrent straight lines are drawn through the vertices A, B, C of a triangle ABC to meet the opposite sides at D, E, F, respectively; then

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=+1.$$

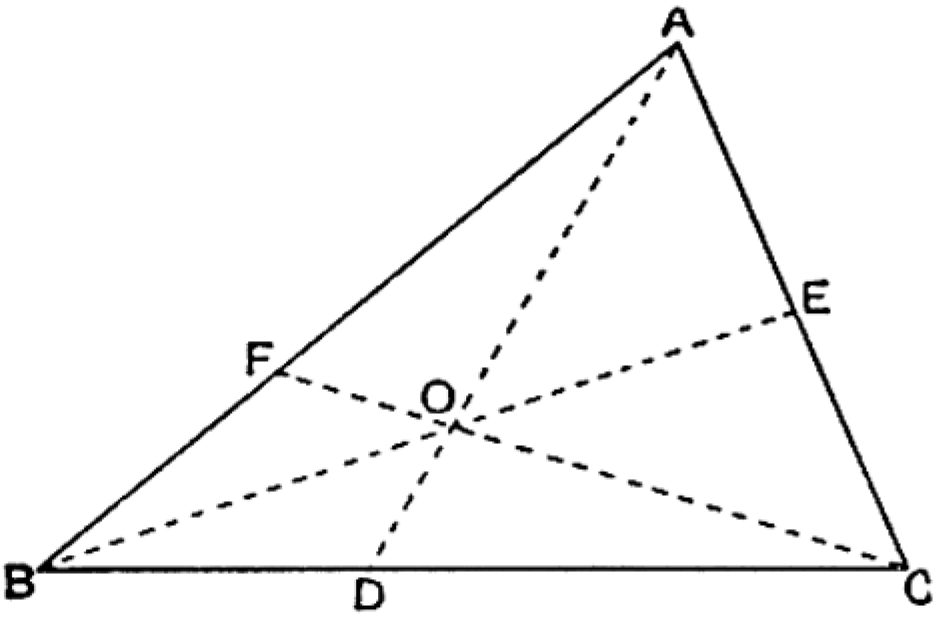

FIG. 29.

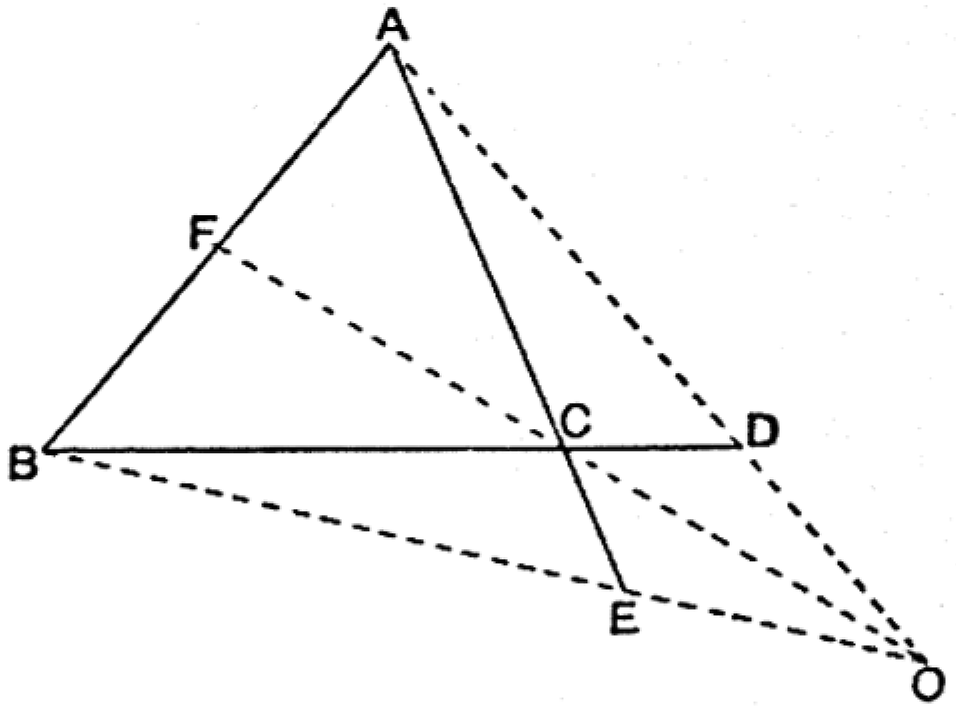

FIG. 30.

Let the three lines concur at O.

If O lies inside the triangle ABC (Fig. 29), each of the three ratios is positive; and if O lies outside the triangle (Fig. 30), one ratio is positive and the other two are negative.

∴ in either case, the product of the three ratios is positive.

By Theorem 5,

$$\frac{BD}{DC}=\frac{\triangle BOA}{\triangle COA},\quad \frac{CE}{EA}=\frac{\triangle BOC}{\triangle BOA},\quad \frac{AF}{FB}=\frac{\triangle COA}{\triangle BOC};$$

$$\therefore\ \frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=+1. \qquad \text{Q.E.D.}$$

Corollary.

If D, E, F are points on BC, CA, AB, such that

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=+1,$$

then AD, BE, CF are concurrent.

[This is easily proved by a 'reductio ad absurdum' method.]

10. If, in Fig. 29, $BD = \frac{1}{2}DC$, $CE = \frac{4}{5}CA$, calculate

$$(1)\ \frac{AF}{FB};\quad (2)\ \triangle AOB : \triangle BOC : \triangle COA;\quad (3)\ \frac{AO}{AD}.$$

11. Points D, E, F are taken on the sides BC, CA, AB of a triangle so that $BD = \frac{1}{3}BC$, $CE = \frac{2}{5}CA$, $AF = \frac{3}{4}AB$; prove that AD, BE, CF are concurrent.

12. In the triangle ABC, BC is produced to D so that BC = CD, CA is produced to E so that CA = AE; if AD meets BE at O and if OC meets AB at F, calculate $\frac{BF}{FA}$.

13. Use Ceva's theorem to prove that

(1) the medians of a triangle are concurrent;
(2) the internal bisectors of the angles of a triangle are concurrent;
(3) the altitudes of a triangle are concurrent.

[Note that $\frac{AF}{AE} = \frac{AC}{AB}$, etc., similar triangles.]

14. If the in-circle of a triangle ABC touches BC, CA, AB at X, Y, Z, prove that AX, BY, CZ are concurrent.

15. In Fig. 29, prove that $\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1$.

16. In Fig. 29, prove that $\frac{AF}{FB} + \frac{AE}{EC} = \frac{AO}{OD}$.

17. Prove that Ceva's test for concurrency can be written as

$$\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin CBE}{\sin EBA} \cdot \frac{\sin ACF}{\sin FCB} = 1.$$

18. If, in Fig. 29, EF is parallel to BC, prove that BD = DC.

19. If, in Fig. 29, the circle DEF cuts BC, CA, AB again at D′, E′, F′, prove that AD′, BE′, CF′ are concurrent.

20. If, in Fig. 29, lines are drawn through the mid-points of BC, CA, AB parallel to AD, BE, CF, prove that they are concurrent.

21. If, in Fig. 29, P, Q, R are the mid-points of EF, FD, DE, prove that AP, BQ, CR are concurrent.

22. If, in Fig. 29, P, Q, R are points on EF, FD, DE, such that DP, EQ, FR are concurrent, prove that AP, BQ, CR are concurrent.

23. A triangle ABC is inscribed in a triangle XYZ and is circumscribed about a triangle PQR; the order of the letters round the triangles is ARBPCQA and XCYAZBX. If AP, BQ, CR are concurrent, and if AX, BY, CZ are concurrent, prove that PX, QY, RZ are concurrent.

24. AP, BQ, CR are three concurrent lines meeting the opposite sides of the triangle ABC at P, Q, R; F, G, H, A′, B′, C′ are the mid-points of AP, BQ, CR, BC, CA, AB; prove that A′F, B′G, C′H are concurrent.

25. A circle is drawn with centre I, the in-centre of ABC; the perpendiculars from I to BC, CA, AB meet this circle at L, M, N; prove that AL, BM, CN are concurrent.

THEOREM 25. [Menelaus' Theorem.]

If a transversal meets the sides BC, CA, AB of a triangle at D, E, F respectively, then

$$\frac{\mathsf{BD}}{\mathsf{DC}}\cdot\frac{\mathsf{CE}}{\mathsf{EA}}\cdot\frac{\mathsf{AF}}{\mathsf{FB}}=-1.$$

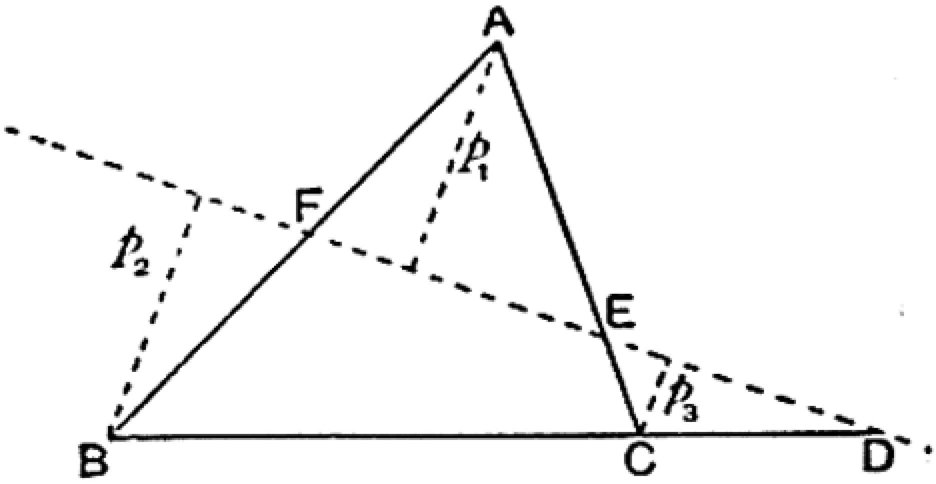

FIG. 31.

The transversal cuts either one side (Fig. 31) or three sides (Fig. 32) of the triangle externally; and therefore either one or three of the ratios $\frac{\mathsf{BD}}{\mathsf{DC}}, \frac{\mathsf{CE}}{\mathsf{EA}}, \frac{\mathsf{AF}}{\mathsf{FB}}$ are negative, and therefore their product is negative in each case.

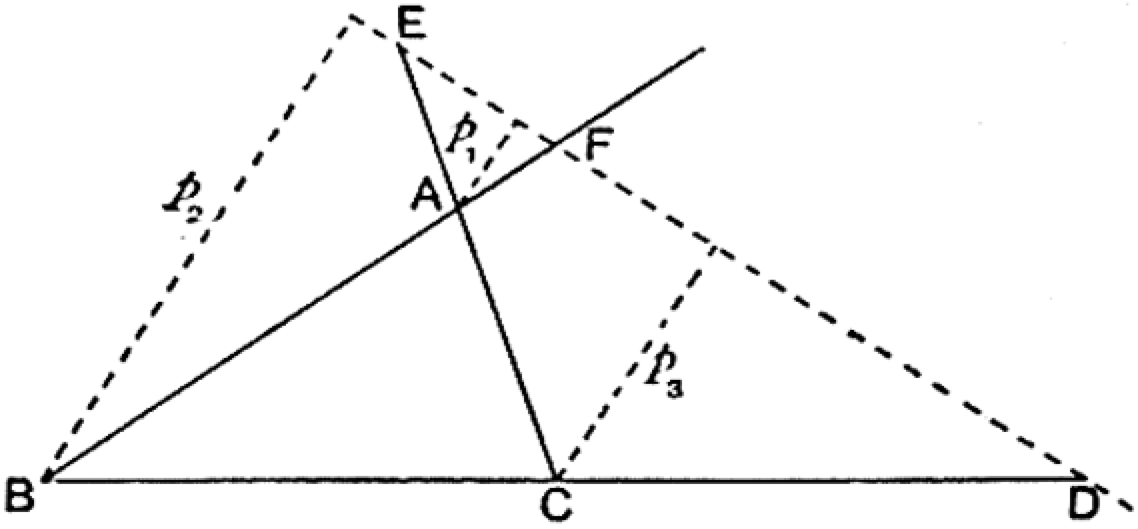

FIG. 32.

From A, B, C drop perpendiculars to the transversal, and let p_1, p_2, p_3 be their lengths.

Then, by similar triangles (Fig. 31),

$$\frac{\mathsf{BD}}{\mathsf{DC}}=-\frac{p_2}{p_3},\quad \frac{\mathsf{CE}}{\mathsf{EA}}=\frac{p_3}{p_1},\quad \frac{\mathsf{AF}}{\mathsf{FB}}=\frac{p_1}{p_2}.$$

$$\therefore\ \frac{\mathsf{BD}}{\mathsf{DC}}\cdot\frac{\mathsf{CE}}{\mathsf{EA}}\cdot\frac{\mathsf{AF}}{\mathsf{FB}}=-1.$$

Q.E.D.

Corollary.

If D, E, F are points on the sides BC, CA, AB of a triangle, such that $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$, then D, E, F are collinear.

This may be easily proved by a 'reductio ad absurdum' method.

26. If, in Fig. 31, BC = 2CD, CA = 3CE, calculate $\frac{AF}{AB}$.

27. B′, C′ are the mid-points of the sides AC, AB of the triangle ABC. Q is the mid-point of B′C′; BQ meets AC at R. Calculate $\frac{AR}{RC}$.

28. If, in Fig. 31, BE, CF are the internal bisectors of the angles ABC, ACB, prove that AD is the external bisector of $B\hat{A}C$.

29. Prove that the points at which the external bisectors of the angles of a triangle meet the opposite sides are collinear.

30. H is a point inside the triangle ABC; prove that the external bisectors of the angles AHB, BHC, CHA meet AB, BC, CA respectively at three collinear points.

31. Points Q, R are taken on the sides CA, AB of a triangle, such that $\frac{CQ}{QA} = \frac{AR}{RB} = k$; QR and CB produced meet at D; prove that $CD = k^2 . BD$.

32. A circle S_1 touches externally each of two circles T_1, T_2, centres A, B, at E, F; another circle S_2 touches externally T_1, T_2 at Q, R; prove that AB, EF, QR are concurrent.

33. G is the centroid of the triangle ABC; AG is produced to P so that GP = AG. Parallels through P to CA, AB, BC meet BC, CA, AB at L, M, N; prove that L, M, N are collinear.

34. The tangents to the circumcircle of a triangle ABC at the vertices A, B, C meet the opposite sides at L, M, N. Prove that L, M, N are collinear.

[Prove that $\frac{BL}{CL} = \frac{BA^2}{CA^2}$, etc.]

35. If, in Fig. 29, EF meets BC at K, prove that $\frac{BD}{DC} = \frac{BK}{CK}$.

36. If, in Fig. 30, EF meets CB at K, prove that $\frac{BD}{DC} = \frac{BK}{CK}$.

37. If, in Fig. 29, EF, FD, DE meet BC, CA, AB at L, M, N, prove that L, M, N are collinear.

38. Four points *a*, *b*, *c*, *d* are taken on the sides AB, BC, CD, DA of a quadrilateral, such that $\frac{Aa}{aB} \cdot \frac{Bb}{bC} \cdot \frac{Cc}{cD} \cdot \frac{Dd}{dA} = 1$; prove that *ab*, *cd* intersect on AC.

39. If a straight line cuts the sides AB, BC, CD, DA of a quadrilateral at P, Q, R, S, prove that $\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = 1$.

40. The in-circle of the triangle ABC touches BC, CA, AB at X, Y, Z; YZ is produced to meet BC at K; prove that $\frac{BX}{XC} = \frac{BK}{CK}$.

41. A straight line meets the sides BC, CA, AB of a triangle at P, Q, R; A′, B′, C′ are the mid-points of these sides; AP, BQ, CR meet B′C′, C′A′, A′B′ at P′, Q′, R′; prove that P′, Q′, R′ are collinear.

42. Three planes AOB, BOC, COA intersect at O. Prove that the plane through the bisectors of the angles AOB, BOC cuts the plane COA in a line bisecting the angle COA externally.

43. If one of the transverse common tangents of two circles is perpendicular to one of the direct common tangents, prove that the eight points of contact lie on two straight lines.

44. [Desargues' Theorem] ABC, A′B′C′ are two triangles, such that AA′, BB′, CC′ meet at O; prove that if BC, B′C′ meet at L; CA, C′A′ at M; AB, A′B′ at N; then L, M, N are collinear.

[Prove $\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = -1$, by considering the transversals LB′C′, MC′A′, NA′B′ applied to the triangles OBC, OCA, OAB.]

45. [Pascal's Theorem] ABCDEF is any hexagon inscribed in a circle; AB, DE meet at L; BC, EF at M; CD, FA at N; prove that L, M, N are collinear.

[Produce AB, CD, EF to form a triangle XYZ, and prove that LMN is a transversal of XYZ by considering BC, DE, FA as transversals of XYZ.]

THEOREM 26.

P, Q, R are points on the sides BC, CA, AB of a triangle, such that the perpendiculars to the sides at these points are concurrent; then

$$BP^2+CQ^2+AR^2=PC^2+QA^2+RB^2.$$

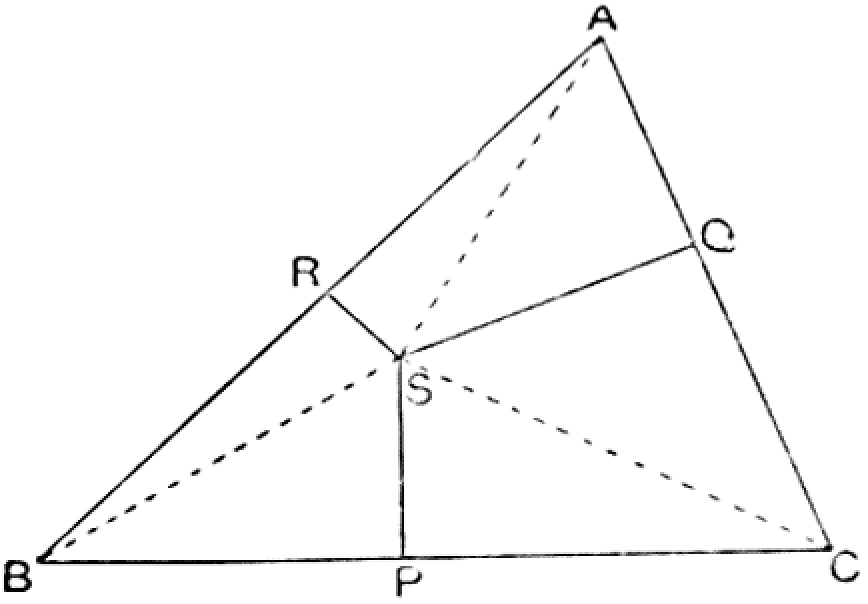

FIG. 33.

Let the perpendiculars at P, Q, R meet at S.

Join SA, SB, SC.

$$BP^2-PC^2=(BP^2+PS^2)-(PC^2+PS^2)=BS^2-SC^2.$$

Similarly $CQ^2-QA^2=CS^2-SA^2$ and $AR^2-RB^2=AS^2-SB^2$.

$\therefore$ by addition, $BP^2-PC^2+CQ^2-QA^2+AR^2-RB^2=0$.

$$\therefore\ BP^2+CQ^2+AR^2=PC^2+QA^2+RB^2. \qquad \text{Q.E.D.}$$

Corollary.

If P, Q, R are points on the sides BC, CA, AB of a triangle, such that $BP^2+CQ^2+AR^2=PC^2+QA^2+RB^2$, then the perpendiculars at P, Q, R to the sides are concurrent.

This may be easily proved by a 'reductio ad absurdum' method.

THEOREM 27. [Simson's Line.]

If P is any point on the circumcircle of the triangle ABC, and if L, M, N are the feet of the perpendiculars from P to BC, CA, AB, then L, M, N are collinear

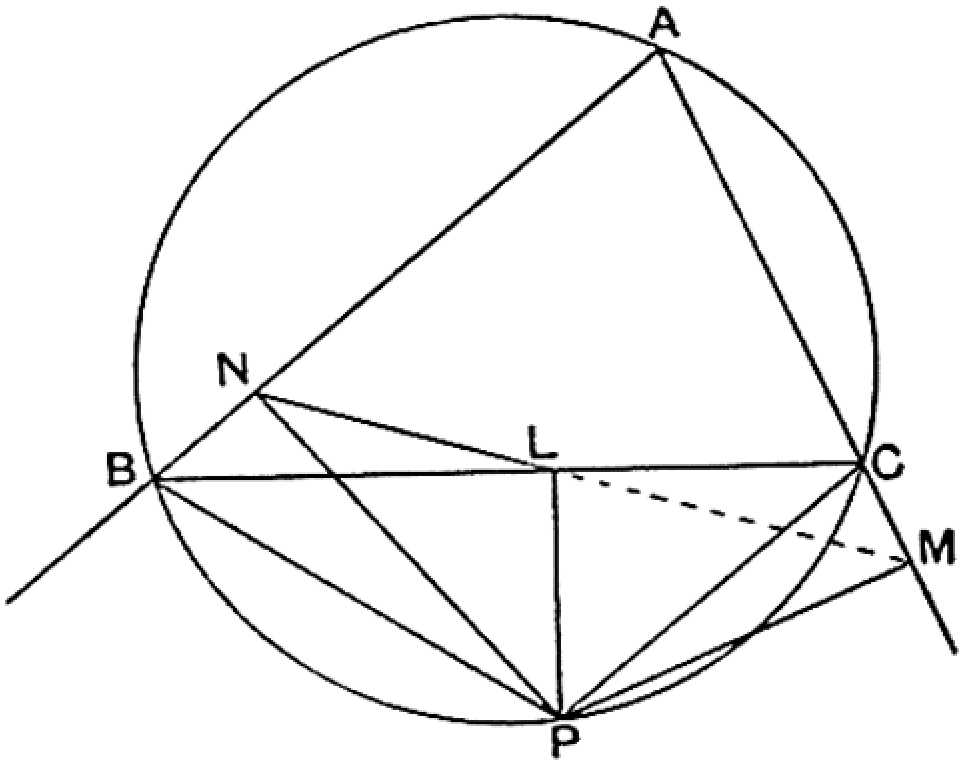

FIG. 34.

Join LN, LM, PB, PC.

Since $\widehat{PNB} = 90° = \widehat{PLB}$, PLNB is a cyclic quadrilateral.

Since $\widehat{PLC} = 90° = \widehat{PMC}$, PLCM is a cyclic quadrilateral.

$\therefore\ \widehat{PLM} = \widehat{PCM}$, since P, L, C, M are concyclic,

$= \widehat{PBA}$, since P, C, A, B are concyclic,

$= 180° - \widehat{PLN}$, since P, L, N, B are concyclic.

$\therefore\ \widehat{PLM} + \widehat{PLN} = 180°$.

$\therefore$ MLN is a straight line. Q.E.D.

Corollary

If the feet of the perpendiculars from a point P to the sides of a triangle ABC are collinear, then P lies on the circumcircle of ABC.

$$[\widehat{PBA} = \widehat{PBN} = 180° - \widehat{PLN} = \widehat{PLM} = \widehat{PCM} = 180° - \widehat{PCA}.]$$

Definition.

The line LMN in Theorem 27 is called the *Simson Line* or the *pedal line* of P w.r.t. the triangle ABC.

46. Use Theorem 26 to prove that the altitudes of a triangle are concurrent.

47. With the usual notation, prove that the perpendiculars from I_1, I_2, I_3 to BC, CA, AB are concurrent.

48. AD, BE, CF are the altitudes of the triangle ABC; prove that the perpendiculars from A, B, C to EF, FD, DE are concurrent.

49. L, M, N are the feet of the perpendiculars from a point K to the sides BC, CA, AB of a triangle; prove that the perpendiculars from A, B, C to MN, NL, LM are concurrent.

50. Each of three circles touches the other two; prove that the common tangents at the points of contact are concurrent.

51. P, Q, R are the feet of the perpendiculars from a point S to the sides BC, CA, AB of a triangle; A′, B′, C′ are the mid-points of these sides; prove that $A'P \cdot BC + B'Q \cdot CA + C'R \cdot AB = 0$.

52. X, Y, Z are the feet of the perpendiculars from the vertices A, B, C of a triangle to any straight line; prove that the perpendiculars from X, Y, Z to BC, CA, AB are concurrent.

53. ABC, XYZ are two triangles such that the perpendiculars from A, B, C to YZ, ZX, XY are concurrent; prove that the perpendiculars from X, Y, Z to BC, CA, AB are concurrent.

54. If the pedal line of P is parallel to AO, prove that PA is parallel to BC.

55. Prove that the pedal line of the point at which AD cuts the circumcircle is parallel to the tangent at A to the circumcircle.

56. In Fig. 34, prove that the triangles PLN, PAC are similar.

57. With the usual notation, prove that the feet of the perpendiculars from A′ to EF, FD, DE are collinear.

58. The perpendiculars from a point P on the circumcircle of the triangle ABC to BC, CA, AB cut the circle again at X, Y, Z; prove that ABC, XYZ are congruent triangles.

59. KL is a diameter of the circumcircle of the triangle ABC; prove that the pedal lines of K and L intersect at right angles on the nine-point circle [use the result of No. 67].

60. P is a point on the circle circumscribing a quadrilateral ABCD; if AB is parallel to CD, prove that the feet of the perpendiculars from P to AC, AD, BC, BD are concyclic.

61. The altitudes AD, BE, CF meet the circumcircle of ABC at P, Q, R; prove that the pedal of A w.r.t. PQR is parallel to BC.

62. P, Q are any two points on the circumcircle of the triangle ABC and O is its circumcentre; prove that the angle between the pedal lines of P and Q equals $\frac{1}{2}P\hat{O}Q$.

63. If, in Theorem 27, PL, PM, PN are drawn so that

$$P\hat{N}B = P\hat{L}B = P\hat{M}C,$$

prove that L, M, N are collinear.

64. Prove that the circumcircles of the four triangles formed by four straight lines have a common point.

65. With the notation of Theorem 27, prove that

$$\frac{PL \,.\, MN}{BC} = \frac{PM \,.\, NL}{CA} = \frac{PN \,.\, LM}{AB}.$$

66. With the notation and figure of Theorem 27, prove that

$$\frac{BC}{PL} = \frac{CA}{PM} + \frac{AB}{PN}.$$

67. If H is the orthocentre of the triangle ABC, prove that the pedal line of P bisects PH.

68. With the notation of Theorem 27, if PK is the perpendicular from P to MN, prove that PM . PN = PA . PK.

69. A chord PQ of the circumcircle of the triangle ABC is parallel to BC; prove that the pedal line of P is perpendicular to AQ.

70. ABC is an equilateral triangle inscribed in a circle, centre O; P is any point on the circumference; prove that the pedal line of P bisects OP

CHAPTER V.

VECTOR GEOMETRY AND STATICAL APPLICATIONS.

CERTAIN quantities, such as velocities, forces, etc., are not completely defined by their numerical magnitude, but involve further the idea of direction. Such quantities can be represented geometrically by a straight line of definite length, drawn in a definite direction, and are called *vector quantities* or *vectors*.

A system of equal and parallel straight lines all represent the same vector. Thus, a velocity of 20 feet per second in a direction N. 30° E. is a vector quantity, and could be represented by any one of a series of equal parallel straight lines. Suppose AB is one of these lines, then the vector AB represents this velocity, and will be denoted by the symbol $\overline{AB}$ in order to avoid confusion between the two ideas, the length of AB and the vector AB.

In order to apply analysis to vector quantities, it is necessary to decide, to start with, what meanings are to be given to the process of addition (or subtraction) and multiplication (or division).

Suppose a particle receives successively two displacements represented in magnitude and direction by $\overline{AB}$ and $\overline{BC}$, the final result is the same as would be the case if it received a single displacement represented by $\overline{AC}$.

By extending the meaning of the word *addition*, we may say that the result of adding the two displacements AB and $\overline{BC}$ is the single displacement $\overline{AC}$.

Definition.

The result of *adding* two vectors $\overline{AB}$ and $\overline{BC}$ together is defined to be the vector $\overline{AC}$; or $\overline{AB}+\overline{BC}=\overline{AC}$.

This is a definition of what is meant by *addition*.

It does not state that the length of AB + the length of BC equals the length of AC, which is untrue unless A, B, C are collinear points; but that the result of adding two quantities, represented in magnitude and direction by AB and BC, is a quantity represented in magnitude and direction by AC.

It is now necessary to show that, with this definition, vectors obey the fundamental laws of Algebra, as regards addition; namely

$$(1)\quad \overline{AB} + \overline{BC} = \overline{BC} + \overline{AB};$$

$$(2)\quad (\overline{AB} + \overline{AC}) + \overline{AD} = \overline{AB} + (\overline{AC} + \overline{AD}).$$

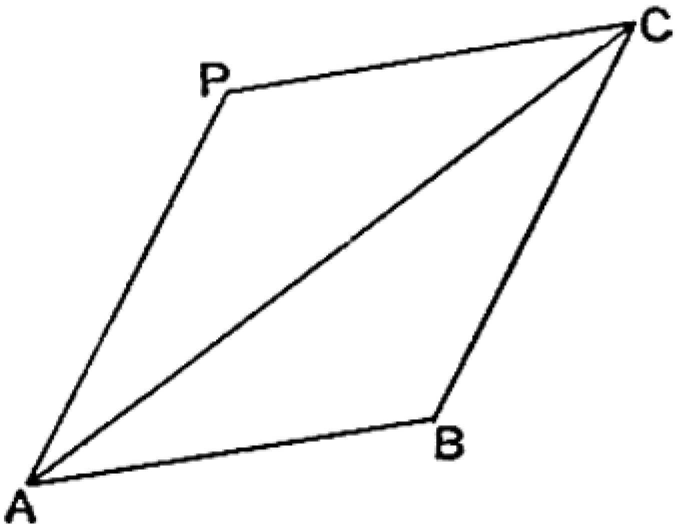

FIG. 35.

Complete the parallelogram ABCP.

By definition, $\overline{AB} + \overline{BC} = \overline{AC} = \overline{AP} + \overline{PC}$.

But $\overline{AP} = \overline{BC}$ and $\overline{PC} = \overline{AB}$.

$\therefore\ \overline{AB} + \overline{BC} = \overline{BC} + \overline{AB}$. Q.E.D.

It is important to note that the sum of two vectors $\overline{AB}$ and $\overline{AP}$ is the vector represented by the diagonal through A of the completed parallelogram PABC.

In Fig. 35, $\overline{AB} + \overline{AP} = \overline{AC}$.

In order to prove (2), complete the parallelograms BACP, DAPQ, so that $\overline{AQ}$ represents the left-hand side: then complete the parallelogram DACR and show that $\overline{AQ} = \overline{AB} + \overline{AR}$.

The process of subtraction may be deduced from that of addition.

For $\overline{AB} + \overline{BA} = \overline{AA} = 0$; $\therefore\ \overline{AB} = -\overline{BA}$.

And $\overline{AB} - \overline{CB} = \overline{AB} + \overline{BC} = \overline{AC}$.

THEOREM 28.

(1) If D is the mid-point of the base BC of a triangle ABC, then

$$\overline{AB} + \overline{AC} = 2\overline{AD}.$$

(2) If D is a point on the base BC of a triangle ABC, such that $\lambda . BD = \mu . DC$, then $\lambda . \overline{AB} + \mu . \overline{AC} = (\lambda + \mu)\overline{AD}$.

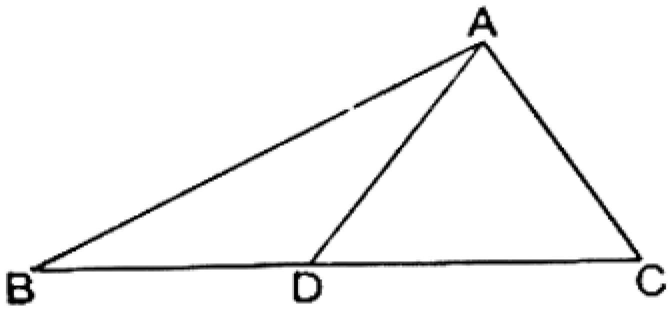

FIG. 36.

(1) is simply a special (although important) case of (2), deduced by putting $\lambda = \mu$. It is therefore only necessary to prove (2).

$$\overline{AB} = \overline{AD} + \overline{DB} \quad \text{and} \quad \overline{AC} = \overline{AD} + \overline{DC}.$$

$$\therefore \lambda . \overline{AB} + \mu . \overline{AC} = (\lambda + \mu) . \overline{AD} + \lambda . \overline{DB} + \mu . \overline{DC}.$$

But $\lambda\overline{DB} + \mu\overline{DC} = 0$, for the senses of DB and DC are opposite and $\lambda . BD = \mu . DC$.

$$\therefore \lambda . \overline{AB} + \mu . \overline{AC} = (\lambda + \mu) . \overline{AD}. \qquad \text{Q.E.D.}$$

Now, from elementary statics, if $\lambda . BD = \mu . DC$, it follows that D is the centre of gravity of masses λ at B and μ at C.

Theorem 28 (2) can therefore be re-written as follows:

If G is the centre of gravity of masses m_1, m_2 at A_1, A_2, and if O is any point,

$$m_1 . \overline{OA_1} + m_2 . \overline{OA_2} = (m_1 + m_2) . \overline{OG}.$$

This result can be readily generalised:

If G is the centre of gravity of any number of masses $m_1, m_2, \dots m_n$ at $A_1, A_2, \dots A_n$, and if O is any point,

$$m_1 . \overline{OA_1} + m_2 . \overline{OA_2} + \dots + m_n . \overline{OA_n} = (m_1 + m_2 + \dots + m_n) . \overline{OG}.$$

In the particular case when the masses are all equal, we have

$$\overline{OG} = \frac{1}{n}(\overline{OA_1} + \overline{OA_2} + \dots + \overline{OA_n}),$$

where G is the centre of gravity of equal masses at $A_1, A_2, \dots A_n$.

This point is also called the *mean centre* or *centroid* of the system of points $A_1, A_2, \dots A_n$.

It is proved in elementary statics that the centre of gravity of a triangular lamina ABC coincides with the mean centre of the points A, B, C. This is not true in the case of a quadrilateral.

As has been already pointed out, forces are included under the category of vector quantities.

Consequently the vector theorem given above can be stated as follows:

If a system of forces, acting at a point O, is represented completely by $m_1 . OA_1, m_2 . OA_2, \dots m_n . OA_n$, and if G is the centroid of masses $m_1, m_2, \dots m_n$ at $A_1, A_2, \dots A_n$, then the resultant of the system is a single force represented completely by

$$(m_1 + m_2 + \dots + m_n) . OG.$$

A similar theorem holds for velocities, couples and all other vector quantities.

1. If $$\overline{PG} = \frac{1}{n}(\overline{PA_1} + \overline{PA_2} + \dots + \overline{PA_n}),$$

and if $$\overline{QH} = \frac{1}{n}(\overline{QA_1} + \overline{QA_2} + \dots + \overline{QA_n}),$$

prove that G coincides with H.

2. If A′, B′, C′ are the mid-points of BC, CA, AB, prove that the mean centre of A′, B′, C′ coincides with the mean centre of A, B, C.

3. Prove that the mean centre of the points A, B, C, D is at the mid-point of the line joining the mid-points of AB and CD. Hence prove that the lines joining the mid-points of opposite sides of a quadrilateral and joining the mid-points of the diagonals are concurrent and bisect each other.

4. If G is the mean centre of ABC, prove that $\overline{GA} + \overline{GB} + \overline{GC} = 0$.

5. If A_1, B_1, C_1, D_1 are the centroids of BCD, CDA, DAB, ABC, prove that AA_1, BB_1, CC_1, DD_1 are concurrent (at G say) and that $AG = 3GA_1$.

6. If A, B, C, D, E, F are the mid-points of consecutive sides of a hexagon, prove that the triangles ACE, BDF have the same centroid.

7. Points P, Q, R are taken on the sides BC, CA, AB of a triangle, such that $\frac{BP}{PC} = \frac{CQ}{QA} = \frac{AR}{RB}$; prove that the triangles ABC, PQR have the same centroid.

8. The sides AB, AC of a triangle are divided at P, Q, so that $\frac{AP}{AB}=\frac{CQ}{CA}$: the parallelograms APRC, ABSQ are drawn; prove that PS and QR are each parallel to the median through A of the triangle ABC.

9. a, b, c are points on OA, OB, OC, such that $\frac{Oa}{OA}=\frac{Ob}{OB}=\frac{Oc}{OC}=k$; A′, B′, C′ are the mid-points of BC, CA, AB; prove that the lines aA′, bB′, cC′ concur at a point P given by $\overline{OP}=\frac{k}{1+2k}(\overline{OA}+\overline{OB}+\overline{OC})$.

10. (1) O is the centre of a circle $A_1A_2A_3$ of unit radius; another equal circle, centre α_1, is drawn through A_2 and A_3; prove that

$$\overline{O\alpha_1}=\overline{OA_2}+\overline{OA_3}.$$

If, further, two other circles of unit radius, centres α_2, α_3, are drawn through A_3, A_1, and A_1, A_2, prove that the points α_1, α_2, α_3 lie on a circle of unit radius (centre β_4 say). [Show that $\overline{O\beta_4}=\overline{OA_1}+\overline{OA_2}+\overline{OA_3}$.]

(2) If four points A_1, A_2, A_3, A_4 lie on a circle of unit radius, and if from every set of three points a circle is constructed as in (1), prove that the centres β_1, β_2, β_3, β_4 of these four circles lie on a circle of unit radius (centre γ_5 say). [Show that $\overline{O\gamma_5}=\overline{OA_1}+\overline{OA_2}+\overline{OA_3}+\overline{OA_4}$.]

(3) If five points A_1, A_2, A_3, A_4, A_5 lie on a circle of unit radius, and if from every four points a circle of unit radius is constructed as in (2), prove that the centres γ_1, γ_2, γ_3, γ_4, γ_5 of these five circles lie on a circle of unit radius (centre δ_6 say): and so on indefinitely.

11. Prove that the lines joining the mid-points of opposite edges of a tetrahedron concur and bisect each other, the point of intersection being the mean centre of the vertices of the tetrahedron.

12. Prove that the lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent and that the point of intersection divides each line in the ratio 3 : 1.

13. If ABCD is a tetrahedron, and if P, Q, R, S are the centroids of the faces BCD, CDA, DAB, ABC, prove that

(i) ABCD and PQRS have the same mean centres,

(ii) PQ is parallel to AB and $PQ=\frac{1}{3}AB$.

14. If X, Y, Z, W are the mid-points of the edges AB, BC, CD, DA of a tetrahedron ABCD, prove that the tetrahedra ABCD, XYZW have the same mean centre.

VECTOR MULTIPLICATION.

If OA represents a unit of length measured in a positive sense along the axis Ox, and if OP represents k units of length, the ratio $\frac{OP}{OA}$ equals k. The position of P depends on the scale chosen, but the ratio $\frac{\mathsf{OP}}{\mathsf{OA}}$ is a pure number (positive or negative) and does not depend on the scale. If then we wish to avoid specifying the unit of length employed, whether inches or cms. or yards, etc., it is necessary to work with ratios; and when, for example, the product OP . OQ is introduced, it is really an abbreviation for $\frac{\mathsf{OP}}{\mathsf{OA}} \cdot \frac{\mathsf{OQ}}{\mathsf{OA}}$.

Given a unit length OA on the x-axis and any real number k, it is possible to find a point P on Ox, such that $\frac{\overline{\mathsf{OP}}}{\overline{\mathsf{OA}}}=k$: and for any point P on OA, the ratio $\frac{\overline{\mathsf{OP}}}{\overline{\mathsf{OA}}}$ represents a real number, positive or negative.

If P is a point outside Ox, we shall extend the *meaning* of *number* and say that the ratio $\frac{\overline{\mathsf{OP}}}{\overline{\mathsf{OA}}}$ is a *complex number*.

If OP $=r$ units of length and $\mathsf{A}\hat{\mathsf{O}}\mathsf{P}=\theta$ radians, $\frac{\overline{\mathsf{OP}}}{\overline{\mathsf{OA}}}$ will also be denoted by (r, θ).

It follows that every positive real number can be denoted by $(r, 0)$ and every negative real number by (r, π), where r is a real positive number: and that the numbers $(r, 0)$, $(r, 2\pi)$, $(r, 4\pi)$, etc., are equivalent.

Again, take two perpendicular axes Ox, Oy. If the coordinates of P are x, y, the number $\frac{\overline{\mathsf{OP}}}{\overline{\mathsf{OA}}}$ will be denoted by $[x, y]$, using square brackets.

Or, more generally, if the projections of PQ on Ox, Oy are h, k, the number $\frac{\overline{\mathsf{PQ}}}{\overline{\mathsf{OA}}}$ will be denoted by $[h, k]$.

It follows at once from a figure

(1) that the definition of vector addition $\overline{\mathsf{OP}}+\overline{\mathsf{PQ}}=\overline{\mathsf{OQ}}$ is equivalent to $[x, y]+[h, k]=[x+h, y+k]$;

(2) that if $[x_1, y_1]=[x_2, y_2]$, then $x_1=x_2$ and $y_1=y_2$, for the vectors are represented by equal and parallel straight lines.

So far, no meaning has been attached to the process of multiplication of vector quantities.

Consider the following arithmetical statements :

(i) $(2, 0) \times (3, 0) = \ \ 2 \times \ \ 3 = \ \ 6 = (6, 0)$;

(ii) $(2, \pi) \times (3, 0) = -2 \times \ \ 3 = -6 = (6, \pi)$;

(iii) $(2, 0) \times (3, \pi) = \ \ 2 \times -3 = -6 = (6, \pi)$;

(iv) $(2, \pi) \times (3, \pi) = -2 \times -3 = \ \ 6 = (6, 0) = (6, 2\pi)$.

These show that, in the case of *real* numbers, the law of multiplication may be stated in the form

$$(r_1, \theta_1) \times (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2).$$

In the case of *complex* numbers, we shall *define multiplication* by the above relation and use it to discover what properties this involves.

GRAPHICAL CONSTRUCTION FOR $(r_1, \theta_1) \times (r_2, \vartheta_2)$.

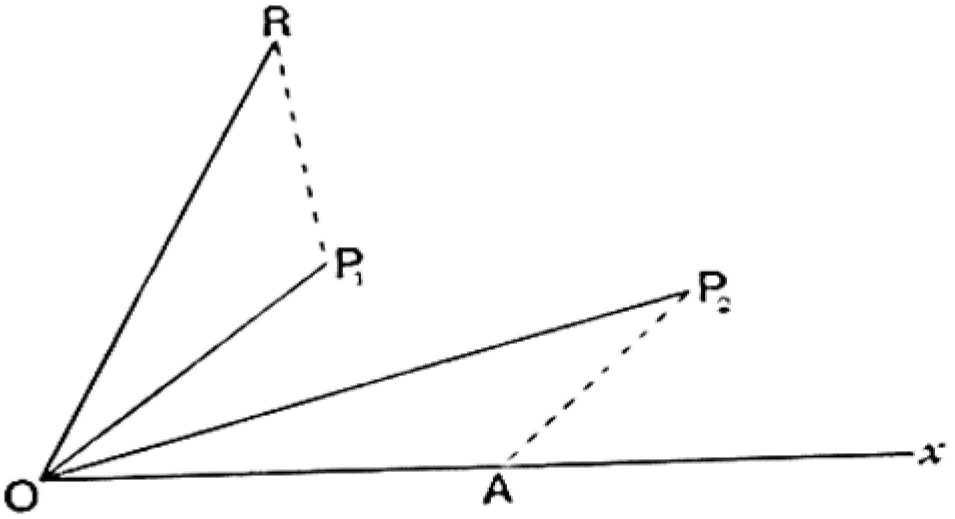

FIG. 37

Let $(r_1, \theta_1) = \frac{\overline{OP_1}}{\overline{OA}}$, $(r_2, \theta_2) = \frac{\overline{OP_2}}{\overline{OA}}$, where OA represents a unit length along Ox.

On OP_1 construct a triangle OP_1R directly similar to OAP_2.

Then $\quad \frac{OR}{OP_2} = \frac{OP_1}{OA}; \quad \therefore \ OR = r_1 r_2,$

and $\quad A\hat{O}R = A\hat{O}P_1 + P_1\hat{O}R = A\hat{O}P_1 + A\hat{O}P_2 = \theta_1 + \theta_2.$

$$\therefore \ \frac{\overline{OR}}{\overline{OA}} = \left(\frac{OR}{OA}, \ A\hat{O}R\right) = (r_1 r_2, \ \vartheta_1 + \theta_2) = (r_1, \theta_1) \times (r_2, \theta_2).$$

Q.E.D

This construction may be stated as follows: in order to multiply $\frac{\overline{\mathrm{OP}_1}}{\overline{\mathrm{OA}}}$ by $\frac{\overline{\mathrm{OP}_2}}{\overline{\mathrm{OA}}}$, the line OP_1 is revolved about O through an angle AOP_2 and is stretched in the ratio $\frac{\mathrm{OP}_2}{\mathrm{OA}}$.

By means of similar triangles, using the above construction, it is easy to show that complex numbers, as defined above, obey the fundamental laws of Algebra, as regards multiplication, namely

(i) $(r_1, \theta_1)\times(r_2, \theta_2)=(r_2, \theta_2)\times(r_1, \theta_1)$.

(ii) $\{(r_1, \theta_1)\times(r_2, \theta_2)\}\times(r_3, \theta_3)=(r_1, \theta_1)\times\{(r_2, \theta_2)\times(r_3, \theta_3)\}$.

(iii) $(r_1, \theta_1)\{(r_2, \theta_2)+(r_3, \theta_3)\}=(r_1, \theta_1)\times(r_2, \theta_2)+(r_1, \theta_1)\times(r_3, \theta_3)$.

INTERPRETATION OF $\frac{\overline{\mathrm{OP}}}{\overline{\mathrm{OA}}}=(r, \theta)=[x, y]$,

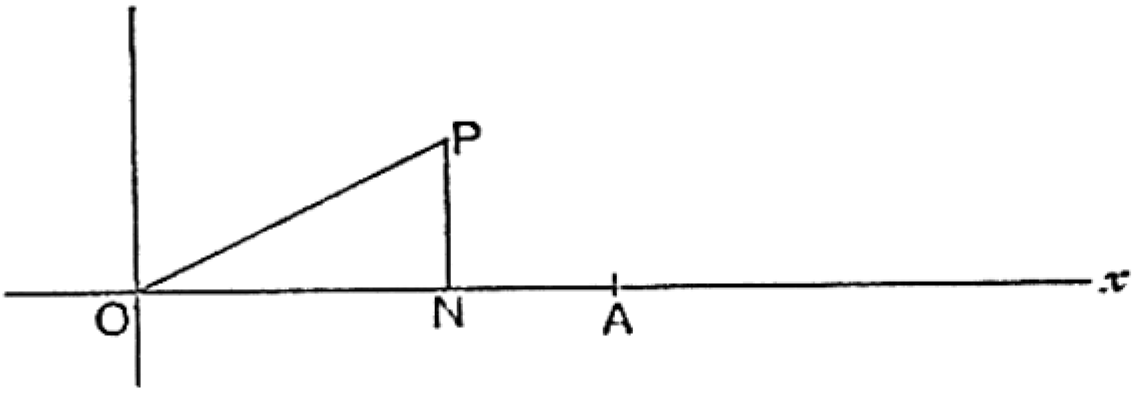

FIG. 38.

By definition, $\left(1, \frac{\pi}{2}\right)\times\left(1, \frac{\pi}{2}\right)=(1, \pi)=-1$.

$\left(1, \frac{\pi}{2}\right)$ may therefore be called the square root of -1, and is denoted by $\sqrt{-1}$ or i.

From P draw PN perpendicular to Ox.

Then
$$\frac{\overline{\mathrm{OP}}}{\overline{\mathrm{OA}}}=\frac{\overline{\mathrm{ON}}+\overline{\mathrm{NP}}}{\overline{\mathrm{OA}}}=\frac{\overline{\mathrm{ON}}}{\overline{\mathrm{OA}}}+\frac{\overline{\mathrm{NP}}}{\overline{\mathrm{OA}}}=(x, 0)+\left(y, \frac{\pi}{2}\right)$$
$$=x+y\left(1, \frac{\pi}{2}\right)=x+y\,.\,i$$
$$=r\cos\theta+r\sin\theta\,.\,i=r(\cos\theta+i\sin\theta).$$
$$\therefore\ (r, \theta)=r(\cos\theta+i\sin\theta)\quad\text{and}\quad[x, y]=x+iy.$$

DEMOIVRE'S THEOREM

If n is any positive integer, then

$$(\cos\theta + i\sin\theta)^n = \cos n\,\theta + i\sin n\theta.$$

$$(1,\ \theta)^n = (1,\ \theta)\times(1,\ \theta)\times\ldots \text{ to } n \text{ factors}$$

$$= (1,\ n\theta) \text{ by the law of multiplication.}$$

$$\therefore\ (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \qquad \text{Q.E.D.}$$

This theorem may be extended to the case in which n is any real number.

For a more detailed discussion, of complex number, the reader is referred to *Hardy's Pure Mathematics*, Ch. III.

15. From the identity $(1,\ \theta)\times(1,\ -\theta)\equiv 1$, deduce that $\cos^2\theta + \sin^2\theta = 1$, and so prove Pythagoras' theorem.

16. From the identity $(1,\ \theta)\times(1,\ \phi)\equiv(1,\ \theta+\phi)$, deduce the expansions of $\sin(\theta+\phi)$ and $\cos(\theta+\phi)$.

17. From $(1,\ \theta)^3\equiv(1,\ 3\theta)$, deduce the expansions of $\sin 3\theta$ and $\cos 3\theta$.

18. Verify geometrically that $(1,\ \theta)+(1,\ -\theta)=2\cos\theta$.

19. Verify geometrically that $(1,\ 0)+\left(1,\ \frac{2\pi}{3}\right)+\left(1,\ \frac{4\pi}{3}\right)=0$.

20. Verify geometrically the following:

$$\text{(i)}\ (1,\ 0)+(1,\ \theta)=\left(2\cos\frac{\theta}{2},\ \frac{\theta}{2}\right);$$

$$\text{(ii)}\ i(1+i) = -1+i;$$

$$\text{(iii)}\ (1-i)^2 = -2i.$$

21. **Prove that $\left(1,\ \frac{\theta}{3}\right)$ is one cube root of $(1,\ \theta)$, and find the other two cube roots.**

22. **Find the three cube roots of $(1,\ \pi)$; represent them graphically; and compare them with the roots of $x^3+1\equiv(x+1)(x^2-x+1)=0$.**

23. **Find the three cube roots of i and of $\frac{1-i}{\sqrt{2}}$.**

24. **AD is a median of the triangle ABC; interpret trigonometrically the relation $\overline{\mathrm{AD}}=\frac{1}{2}(\overline{\mathrm{AB}}+\overline{\mathrm{AC}})$.**

25. **Prove that the vector equation $\overline{\mathrm{BC}}=\overline{\mathrm{BA}}+\overline{\mathrm{AC}}$ is equivalent to the two formulae: $a=b\cos\mathrm{C}+c\cos\mathrm{B}$; $\frac{b}{\sin\mathrm{B}}=\frac{c}{\sin\mathrm{C}}$.**

26. If AD is a median of the triangle ABC, prove that

$$\overline{AB}^2+\overline{AC}^2=2\overline{AD}^2+2\overline{DB}^2.$$

27. If ABCD is a plane quadrilateral, prove that

$$\overline{AC}\,.\,\overline{BD}=\overline{AB}\,.\,\overline{CD}+\overline{BC}\,.\,\overline{AD}.$$

By expressing this result trigonometrically, prove that if A, B, C, D are concyclic, then AC . BD = AB . CD + BC . AD; but that otherwise AC . BD < AB . CD + BC . AD.

28. If λ, μ, ν are real numbers, such that $\lambda+\mu+\nu=0$, and if A, B, C are points, such that $\lambda\,.\,\overline{OA}+\mu\,.\,\overline{OB}+\nu\,.\,\overline{OC}=0$, prove that A, B, C are collinear.

If G is the centre of gravity of masses $m_1, m_2, \dots m_n$ situated at $A_1, A_2, \dots A_n$, whose coordinates are $(x_1, y_1), (x_2, y_2), \dots (x_n, y_n)$, and if $(\bar{x}, \bar{y})$ are the coordinates of G, the theorem

$$(m_1+m_2+\dots+m_n)\overline{OG}=m_1\,.\,\overline{OA}_1+m_2\,.\,\overline{OA}_2+\dots+m_n\,.\,\overline{OA}_n$$

can be written

$$(m_1+m_2+\dots+m_n)(\bar{x}+i\bar{y})$$
$$=m_1(x_1+iy_1)+m_2(x_2+iy_2)+\dots+m_n(x_n+iy_n),$$

or
$$(m_1+\dots+m_n)\bar{x}=m_1x_1+\dots+m_nx_n$$

and
$$(m_1+\dots+m_n)\bar{y}=m_1y_1+\dots+m_ny_n.$$

The centre of gravity is therefore determined by the equations

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}, \quad \bar{y}=\frac{\Sigma my}{\Sigma m}.$$

These results also follow of course from taking moments about Oy and Ox for the system.

THEOREM 29.

If O is the centre of gravity of masses m_1, m_2, m_3 at A, B, C, then

$$\frac{m_1}{\triangle \mathrm{OBC}} = \frac{m_2}{\triangle \mathrm{OCA}} = \frac{m_3}{\triangle \mathrm{OAB}}.$$

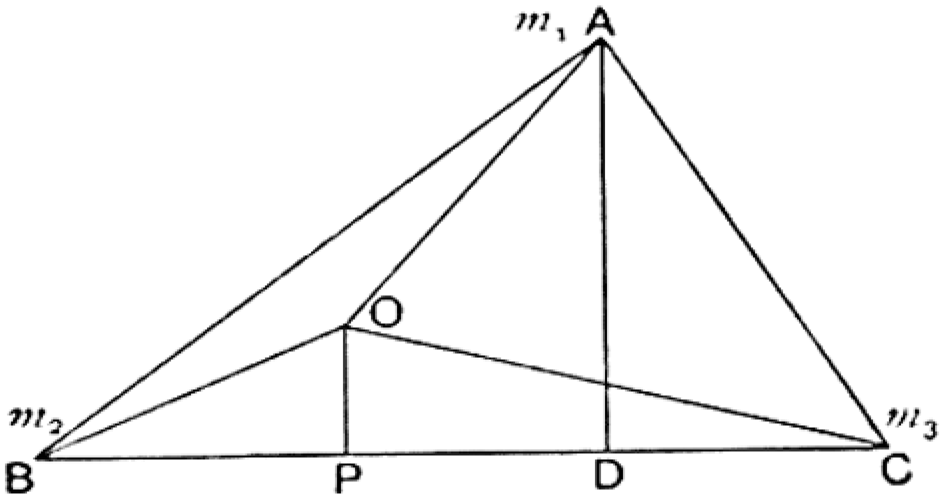

FIG. 39.

Draw OP, AD perpendicular to BC.

Regard BC as the x-axis.

$$\therefore\ \mathrm{OP} = \frac{m_1 \,.\, \mathrm{AD} + m_2 \,.\, \mathrm{O} + m_3 \,.\, \mathrm{O}}{m_1 + m_2 + m_3} = \frac{m_1 \,.\, \mathrm{AD}}{m_1 + m_2 + m_3};$$

$$\therefore\ \frac{m_1}{\mathrm{OP}} = \frac{m_1 + m_2 + m_3}{\mathrm{AD}} \quad \text{or} \quad \frac{m_1}{\frac{1}{2}\mathrm{OP} \,.\, \mathrm{BC}} = \frac{m_1 + m_2 + m_3}{\frac{1}{2}\mathrm{AD} \,.\, \mathrm{BC}};$$

$$\therefore\ \frac{m_1}{\triangle \mathrm{OBC}} = \frac{m_1 + m_2 + m_3}{\triangle \mathrm{ABC}} = \text{similarly } \frac{m_2}{\triangle \mathrm{OCA}}, \text{ etc.} \qquad \text{Q.E.D.}$$

In the following examples, $\overline{\mathrm{PQ}}$ denotes the force represented in magnitude and position by PQ.

29. If G is the mean centre of $A_1, A_2, \dots A_n$, and if H is the mean centre of $B_1, B_2, \dots B_m$, and if all the points are collinear, prove that

$$\Sigma\Sigma \mathrm{AB} = mn \,.\, \mathrm{GH}.$$

30. If G is the mean centre of $A_1, A_2, \dots A_n$, and if H is the mean centre of $B_1, B_2, \dots B_n$, and if all the points are collinear, prove that

$$\Sigma A_r B_r = n \,.\, \mathrm{GH}.$$

31. Determine the masses to be placed at A, B, C in order that their centroid may be (1) the incentre, (2) the orthocentre, (3) the circumcentre, (4) the excentre opposite to A.

32. A′, B′, C′ are the mid-points of BC, CA, AB; prove that $\overline{\mathrm{PA}}$, $\overline{\mathrm{PB}}$, $\overline{\mathrm{PC}}$, $\overline{\mathrm{A'P}}$, $\overline{\mathrm{B'P}}$, $\overline{\mathrm{C'P}}$ are in equilibrium.

33. If ABCD is a parallelogram, prove that $\overline{PA}$, $\overline{PC}$, $\overline{BP}$, $\overline{DP}$ are in equilibrium.

34. O, N are the circumcentre, nine-point centre of the triangle ABC; prove that the resultant of $\overline{NA}$, $\overline{NB}$, $\overline{NC}$ lies along the same line as the resultant of $\overline{OA}$, $\overline{OB}$, $\overline{OC}$.

35. Forces P, Q, R, S act along AB, BC, CD, DA and are in equilibrium; prove that $\frac{P \cdot R}{AB \cdot CD} = \frac{Q \cdot S}{BC \cdot DA}$.

36. Find the point P inside the quadrilateral ABCD, such that $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, $\overline{PD}$ are in equilibrium.

37. By considering the resultant of the forces $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, $\overline{PD}$, prove that the straight lines joining the mid-points of opposite edges of the tetrahedron ABCD are concurrent and bisect each other.

38. If ABCD is a quadrilateral, and if $l \cdot \overline{AB}$, $m \cdot \overline{BC}$, $n \cdot \overline{CD}$, $r \cdot \overline{DA}$ are in equilibrium, prove that $ln = mr$.

39. ABCD is a quadrilateral: AD meets BC at E; AB meets DC at F; prove that the resultant of the forces $\overline{AB}$, $\overline{AD}$, $\overline{CB}$, $\overline{CD}$ passes through the mid-points of AC, BD, EF, so proving these points are collinear.

40. $A_1A_2 \dots A_n$ and $B_1B_2 \dots B_m$ are two polygons; find the resultant of all the forces $\overline{A_rB_s}$.

41. Find a point P in the plane of the triangle ABC, such that the following sets of forces are in equilibrium:

(1) $a \cdot \overline{PA}$ $b \cdot \overline{PB}$, $c \cdot \overline{PC}$.
(2) $a \cdot \overline{PA}$, $b \cdot \overline{PB}$, $c \cdot \overline{CP}$.
(3) $\sin 2A \cdot \overline{PA}$, $\sin 2B \cdot \overline{PB}$, $\sin 2C \cdot \overline{PC}$.

42. Three forces P, Q, R act along the edges AB, AC, AD of a tetrahedron; if their resultant is parallel to the face BCD, prove that

$$\frac{P}{AB} + \frac{Q}{AC} + \frac{R}{AD} = 0.$$

43. ABCD is a tetrahedron; I_1, I_2, I_3, I_4 are the incentres of the faces opposite to A, B, C, D; if AI_1, BI_2 intersect, prove that

(1) AD . BC = AC . BD,
(2) CI_3 and DI_4 also intersect,

and if AD . BC = AC . BD = AB . CD, prove that AI_1, BI_2, CI_3, DI_4 are concurrent.

44. Three forces P, Q, R act along the sides BC, CA, AB of the triangle ABC; if their resultant is along IG, prove that

$$P : Q : R = a(b-c) : b(c-a) : c(a-b).$$

THEOREM 30. [Apollonius' Theorem.]

(1) If D is the mid-point of the base BC of a triangle ABC, then

$$AB^2 + AC^2 = 2AD^2 + 2BD^2.$$

(2) If D is a point on the base BC of a triangle ABC, such that $m \cdot BD = n \cdot DC$, then

$$m \cdot AB^2 + n \cdot AC^2 = (m+n)AD^2 + m \cdot DB^2 + n \cdot DC^2.$$

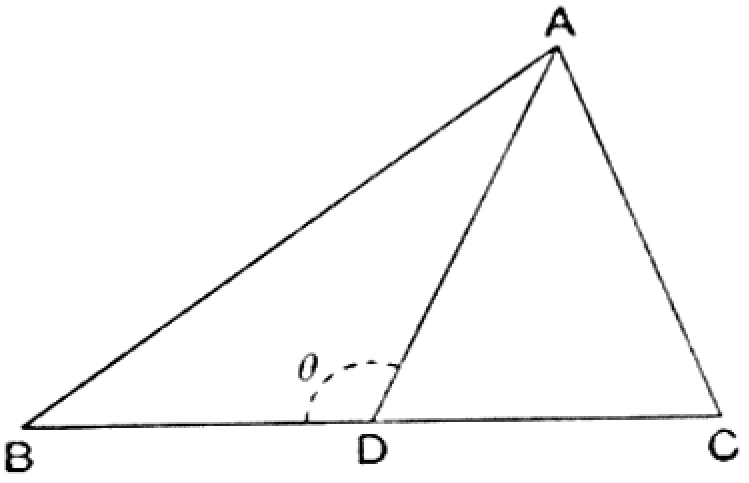

FIG. 40.

(1) Let $\hat{ADB} = \theta; \quad \therefore \hat{ADC} = 180° - \theta.$

$$\therefore AB^2 = AD^2 + DB^2 - 2AD \cdot DB \cos\theta \quad \text{...................(i)}$$

and $$AC^2 = AD^2 + DC^2 + 2AD \cdot DC \cos\theta \quad \text{............ ... (ii)}$$

$\therefore$ by addition, since $BD = DC$,

$$AB^2 + AC^2 = 2AD^2 + 2BD^2.$$ Q.E.D.

(2) Multiply (i) by m and (ii) by n, and add. Since $m \cdot BD = n \cdot DC$,

$$m \cdot AB^2 + n \cdot AC^2 = (m+n)AD^2 + m \cdot DB^2 + n \cdot DC^2.$$

Q.E.D.

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THEOREM 31.

G is the mean centre of the n points $A_1, A_2, \dots A_n$; if P is any other point, then

$$\sum_{r=1}^{n} PA_r^2 - \sum_{r=1}^{n} GA_r^2 = n \, . \, PG^2.$$

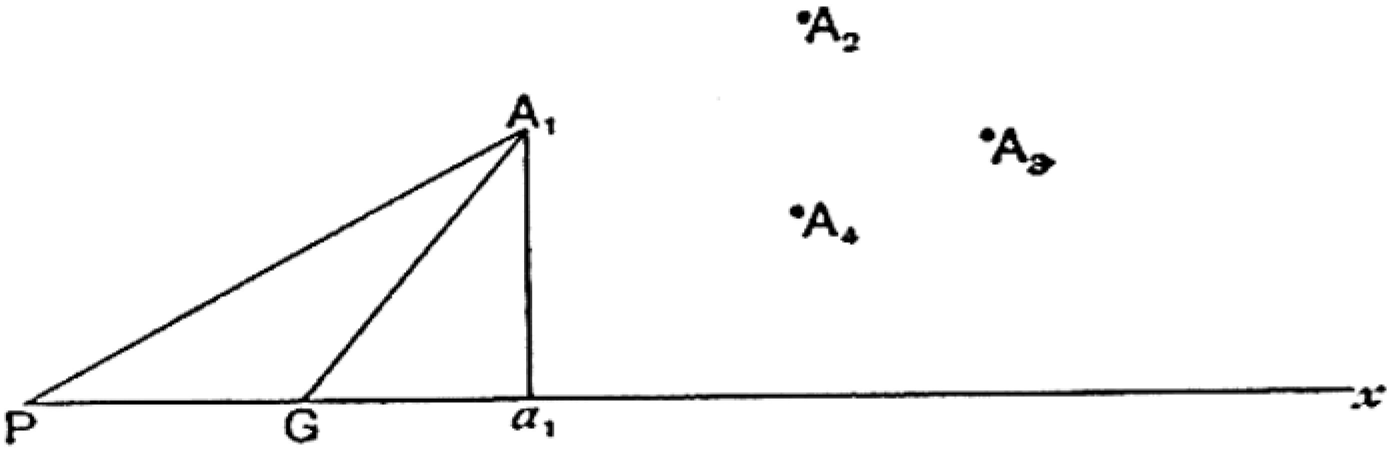

FIG. 41.

Join PG: let $a_1, a_2, a_3 \dots$ be the feet of the perpendiculars from $A_1, A_2, A_3, \dots$ to PG.

Taking G as the origin and PG as the x-axis,

$$\frac{1}{n}(Ga_1 + Ga_2 + \dots + Ga_n) = x\text{-coordinate of } G = 0.$$

$$\therefore \sum_{r=1}^{n} Ga_r = 0.$$

$$\begin{aligned} PA_1^2 - GA_1^2 &= (Pa_1^2 + a_1A_1^2) - (Ga_1^2 + a_1A_1^2) \\ &= (PG + Ga_1)^2 - Ga_1^2 \\ &= PG^2 + 2PG \, . \, Ga_1. \end{aligned}$$

$$\begin{aligned} \therefore \ \Sigma PA_r^2 - \Sigma GA_r^2 &= n \, . \, PG^2 + 2PG \, . \, \Sigma Ga_r \\ &= n \, . \, PG^2. \end{aligned}$$

Q.E.D.

THEOREM 32.

If G is the centroid of masses $m_1, m_2, \dots m_r$ at the n points $A_1, A_2, \dots A_n$, and if P is any other point, then

$$\sum_{r=1}^{n} m_r \, . \, PA_r^2 - \sum_{r=1}^{n} m_r \, . \, GA_r^2 = (m_1 + m_2 + \dots + m_n) \, . \, PG^2.$$

This is proved in exactly the same way as Theorem 31.

Note that Theorem 30 (1) and (2) are special cases of Theorems 31, 32.

45. Calculate the length of the shortest median of a triangle whose sides are 8 cms., 9 cms., 10 cms.

46. Calculate the length of a side of a triangle, whose medians are of lengths x, y, z.

47. The median AA′ of the triangle ABC is a mean proportional between AB, AC; prove that $a = \sqrt{2}(b \sim c)$.

48. Prove that three times the sum of the squares of the sides of a triangle equals four times the sum of the squares of the medians.

49. If ABCD is a parallelogram, prove that

$$AC^2 + BD^2 = 2AB^2 + 2BC^2.$$

50. P, Q are the mid-points of the diagonals AC, BD of the quadrilateral ABCD; prove that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2.$$

51. If the sum of the squares of each pair of opposite edges of a tetrahedron is the same, prove that the lines joining the mid-points of opposite edges are equal.

52. ABCD is a rectangle, and P any point in space; prove that

$$PA^2 + PC^2 = PB^2 + PD^2.$$

53. B is a fixed point, A is the centre of a fixed circle; prove that the locus of the mid-points of chords which subtend a right angle at B is a circle whose centre is the mid-point of AB.

54. The base BC of the triangle ABC is trisected at X, Y; prove that

$$AB^2 + AC^2 = AX^2 + AY^2 + 4XY^2.$$

55. B is a fixed point: A is the centre of a fixed circle: PQ is a variable chord parallel to AB; prove that $BP^2 + BQ^2$ is constant.

56. AP, BQ, CR are the tangents from three of the vertices of the rectangle ABCD to a given circle; if $AP^2 + CR^2 = BQ^2$, prove that D lies on the circle.

57. Squares are described externally on the sides of a triangle and their adjacent corners are joined; prove that the sum of the squares on the joining lines is equal to three times the sum of the squares on the sides of the triangle.

58. In the triangle ABC, AB = 8, AC = $5\frac{3}{4}$, BC = 9; D is a point on BC, such that BD = 4; prove that AD = $5\frac{1}{2}$.

59. A point D divides the base BC of the triangle ABC in the ratio $\lambda : \mu$; AD is produced to meet the circumcircle at E; prove that

$$\mu \,.\, AB^2 + \lambda \,.\, AC^2 = (\mu + \lambda)\, AD \,.\, AE.$$

What does this theorem become if AD (1) is a median, (2) bisects $B\hat{A}C$?

60. ABC is an equilateral triangle, AB is produced to D, so that $BD = \lambda \,.\, AB$; P is any point, such that DP = DC; prove that

$$\lambda \,.\, AP^2 + AB^2 = (\lambda + 1)\, BP^2.$$

61. If AD bisects $B\hat{A}C$ and meets BC at D, prove that

$$AD^2 = bc\left\{1 - \left(\frac{a}{b+c}\right)^2\right\}.$$

62. ABC is a given triangle; find the position of a point P, such that $PA^2 + PB^2 + PC^2$ has its least value.

63. A, B, C are fixed points; P is a variable point, such that

$$PA^2+2PB^2+3PC^2$$

is constant; prove that the locus of P is a circle.

64. If N, H are the nine-point centre and orthocentre of the triangle ABC, prove that N is the mean centre of A, B, C, H.

65. With the usual notation for the triangle ABC, prove that

$$(1)\ AN^2+BN^2+CN^2+9GN^2=3OA^2.$$
$$(2)\ AH^2+BH^2+CH^2=3OA^2+OH^2.$$

66. ABCD is a square; P is a variable point, such that

$$PA^2+PB^2+PC^2+PD^2=3AB^2;$$

prove that the locus of P is a circle whose radius equals $\frac{1}{2}$AB.

67. P is a point on the circumcircle of the equilateral triangle ABC; prove that $PA^2+PB^2+PC^2=2AB^2$.

68. G is the mean centre of n points $A_1, A_2, \dots A_n$; P, Q are any two points; prove that $\Sigma(PA_r^2-QA_r^2)=n(PG^2-QG^2)$.

69. A regular pentagon $A_1A_2A_3A_4A_5$ is inscribed in a circle, centre O; prove that $A_1A_2^2+A_1A_3^2=5A_1O^2$.

70. $A_1A_2 \dots A_n$ is a regular polygon inscribed in a circle, centre O; P is a point on the circumference, Q is any other point; prove that

$$(1)\ \Sigma PA_r^2=2n\,.\,PO^2,$$
$$(2)\ \Sigma QA_r^2=n(PO^2+QO^2),$$
$$(3)\ \Sigma\Sigma A_rA_s^2=n^2\,.\,PO^2.$$

71. Prove that, if regular figures of 5, 6, 10 sides are inscribed in a circle, the square on a side of the pentagon is equal to the sum of the squares on a side of each of the other two figures.

72. $A_1A_2 \dots A_n$ and $B_1B_2 \dots B_n$ are regular polygons inscribed in circles, centres C, D; prove that

$$\Sigma\Sigma A_rB_s^2=n^2(A_1C^2+B_1D^2+CD^2).$$

73. ABC is a triangle; find the position of P for which

$$BC\,.\,PA^2+CA\,.\,PB^2+AB\,.\,PC^2$$

has its least value.

74. G is the centroid of masses l, m, n at A, B, C; GT is the tangent from G to the circumcircle of the triangle ABC; prove that

$$l\,.\,PA^2+m\,.\,PB^2+n\,.\,PC^2=(l+m+n)(PG^2-GT^2).$$

[It is supposed that a negative mass may exist.]

75. If A, B, C are fixed points and P varies, so that

$$\lambda\,.\,PA^2+\mu\,.\,PB^2+\nu\,.\,PC^2=0,$$

prove that the locus of P is a circle cutting the circle ABC at right angles.

76. $A_1A_2 \dots A_n$ is a regular polygon inscribed in a circle, centre O; from any point P, perpendiculars $PN_1, PN_2, \dots$ are drawn to $OA_1, OA_2, \dots$; prove that $\Sigma PN_1^2=\Sigma ON_1^2$.

77. A, B, C are any fixed points, P is a variable point, such that $PA^2-3PB^2+2PC^2$ is constant; find the locus of P.

Consider also the special case where A, B, C are collinear and AB = 2BC.

78 Discuss what modification is required in Theorem 32, if $\Sigma m_r=0$.

CHAPTER VI.

HARMONIC RANGES AND PENCILS.

Definition.

A straight line AB is divided internally at C and externally at D in the same ratio $\left[i.e.\ \frac{AC}{CB} = -\frac{AD}{DB}\right]$; then AB is said to be divided harmonically at C and D; and the points ACBD are said to form a *harmonic range.*

C and D are called *harmonic conjugates* with respect to A and B.

The range ACBD is denoted by {ACBD} or {AB ; CD}.

THEOREM 33.

(1) If C, D divide AB harmonically, then A, B divide CD harmonically.

(2) If O is the mid-point of AB, then $OB^2 = OC \cdot OD$.

FIG. 42.

(1) Since {ACBD} is harmonic, $\frac{AC}{CB} = \frac{AD}{BD}$.

$$\therefore \frac{CB}{BD} = \frac{AC}{AD} = -\frac{CA}{AD}.$$

∴ CD is divided internally at B and externally at A in the same ratio. Q.E.D

(2) Since $\frac{AC}{CB} = \frac{AD}{BD}$, $\therefore \frac{AC+CB}{AC-CB} = \frac{AD+BD}{AD-BD}$.

But

$$AC + CB = AB = 2OB.$$
$$AC - CB = (AO + OC) - (OB - OC) = 2OC.$$
$$AD + BD = (AO + OD) + (OD - OB) = 2OD.$$
$$AD - BD = AB = 2OB.$$
$$\therefore \frac{2OB}{2OC} = \frac{2OD}{2OB} \quad \text{or} \quad OB^2 = OC\,.\,OD. \qquad \text{Q.E.D.}$$

Corollary.

If O is the mid-point of AB, and if C, D are points on AB, such that $OB^2 = OC\,.\,OD$, then {ACBD} is harmonic.

This follows by reversing the order of the steps in (2).

THEOREM 34.

If four concurrent lines are such that one transversal is cut by them harmonically, then every transversal will be cut harmonically by the four lines.

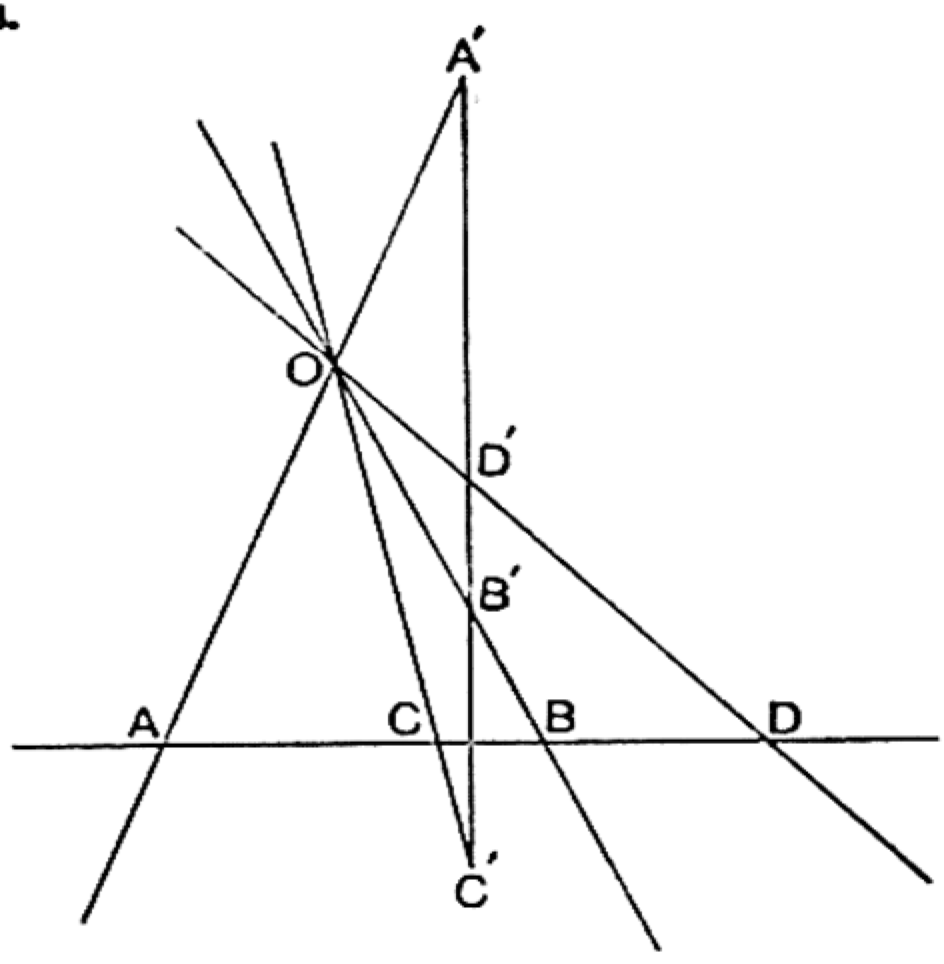

Fig. 43.

Let a line ACBD cut the four lines OA, OC, OB, OD, such that {ACBD} is harmonic.

Let any other line cut the four lines at A′, C , B′, D′.

It is required to prove that {A′C′B′D′} is harmonic.

By Theorem 8,

$$\frac{AC}{CB}=\frac{OA}{OB}\cdot\frac{\sin AOC}{\sin COB} \quad\text{and}\quad \frac{AD}{BD}=\frac{OA}{OB}\cdot\frac{\sin AOD}{\sin BOD}.$$

But $$\frac{AC}{CB}=\frac{AD}{BD};\quad \therefore\ \frac{\sin AOC}{\sin COB}=\frac{\sin AOD}{\sin BOD}.$$

Now $$\frac{A'C'}{C'B'}=\frac{OA'}{OB'}\cdot\frac{\sin A'OC'}{\sin C'OB'} \quad\text{and}\quad \frac{A'D'}{B'D'}=\frac{OA'}{OB'}\cdot\frac{\sin A'OD'}{\sin B'OD'}.$$

But $$\sin AOC=\sin A'OC',\ \text{etc.},$$

these angles being equal or supplementary;

$$\therefore\ \frac{A'C'}{C'B'}=\frac{A'D'}{B'D'};$$

$\therefore$ $\{A'C'B'D'\}$ is harmonic. Q.E.D.

Definition.

Four concurrent lines which cut one (and $\therefore$ every) transversal in a harmonic range are said to form a *harmonic pencil*: and the point of concurrency of the four lines is called the *vertex* of the pencil: each of the lines OA, OC, OB, OD are called *rays* of the pencil: OC, OD are called *conjugate rays* w.r.t. OA, OB. The pencil is represented by O{ACBD} or O{AB ; CD}.

It should be noted that the condition that a pencil is harmonic can be expressed in the form

$$\frac{\sin AOC\ .\ \sin BOD}{\sin COB\ .\ \sin AOD}=1.$$

THEOREM 35.

(1) If O{ACBD} is harmonic, and if a line through B parallel to OA cuts OC, OD at P, Q, then PB = BQ.

(2) If O{ACBD} is harmonic, and if $A\hat{O}B = 90°$, then AO, BO are the bisectors of the angle COD.

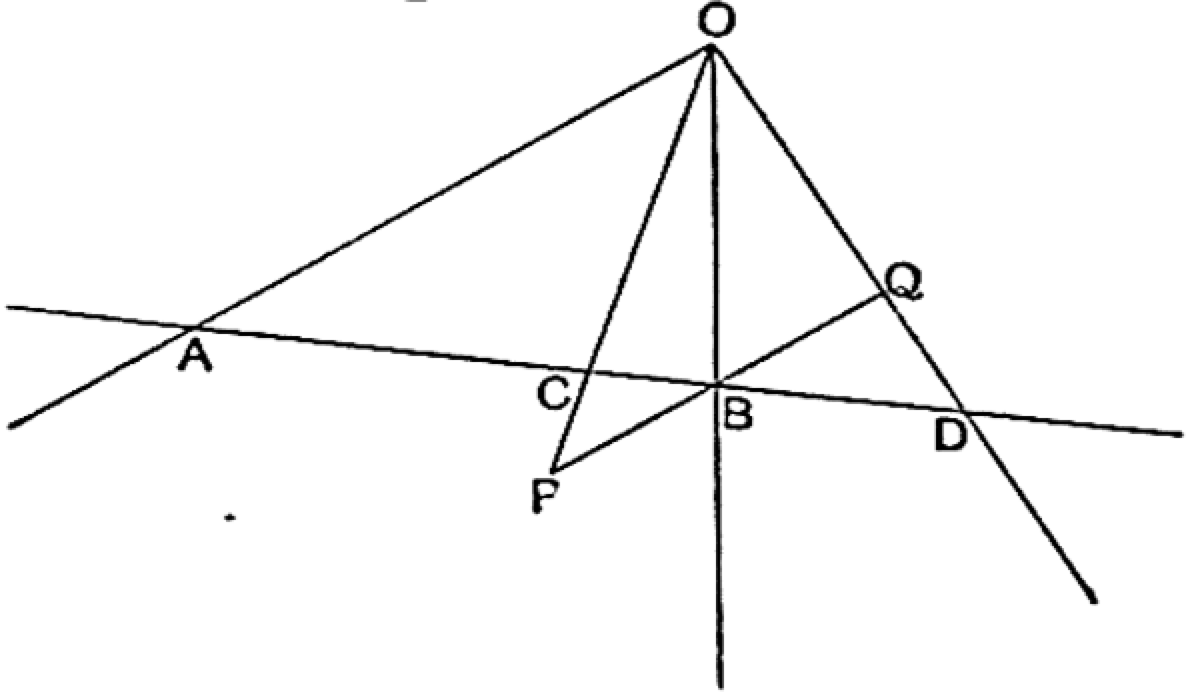

FIG. 44.

(1) Since AO is parallel to PQ, $\frac{AC}{CB} = \frac{AO}{PB}$.

Since AO is parallel to BQ, $\frac{AD}{BD} = \frac{AO}{BQ}$.

But $\frac{AC}{CB} = \frac{AD}{BD}$; $\therefore$ PB = BQ. **Q.E.D.**

(2) Draw PBQ parallel to AO, cutting OC, OD at P, Q.

$\therefore$ $O\hat{B}Q = B\hat{O}A = 90°$, by parallels.

Also PB = BQ from (1).

$\therefore$ the triangles OBP, OBQ are congruent.

$\therefore$ OB bisects $P\hat{O}Q$.

But OA is perpendicular to OB; $\therefore$ OA bisects $P\hat{O}Q$ externally. **Q.E.D.**

Corollary.

If {ACBD} is harmonic, and if P is any point on the circle on CD as diameter, then $\frac{PA}{PB}$ is constant and equal to $\frac{AC}{CB}$.

For, by (2), PC and PD bisect $A\hat{P}B$.

This is the converse of Theorem 9 [Apollonius' Circle].

It should be noted that Theorem 35 (1) is really a special case of Theorem 34, for the line QBP cuts OA at infinity, and $\{QBP\infty\}$ is therefore a harmonic range, where ∞ denotes the point at infinity on QP (see page 38).

$$\therefore \frac{QB}{BP}=\frac{Q\infty}{P\infty}=1; \quad \text{or } QB=BP.$$

1. If $\{ACBD\}$ is harmonic, prove that AC, AB, AD are in harmonical progression.

2. If $\{AB ; CD\}$ is harmonic, and if P is any other point on AD, prove that $\frac{2PB}{AB}=\frac{PC}{AC}+\frac{PD}{AD}$.

3. If $\{ACBD\}$ is harmonic, and if O is the mid-point of AB, prove that

$$(1)\ DA \,.\, DB=DC \,.\, DO,$$
$$(2)\ AB \,.\, CD=2AD \,.\, CB=2AC \,.\, BD.$$

4. If $O\{AB ; CD\}$ is harmonic, prove that

$$2\cot AOB=\cot AOC+\cot AOD.$$

5. Two circles intersect at B, C; a common tangent touches them at P, Q and cuts any other circle through B, C at L, M; prove that $\{PQ ; LM\}$ is harmonic.

6. $\{ACBD\}$ is harmonic; O, O′ are the mid-points of AB, CD; the circles on AB, CD as diameters intersect at H; prove that

$$O\hat{H}O'=90°.$$

7. P, Q, R are points on BC, CA, AB such that AP, BQ, CR are concurrent; P′, Q′, R′ are the harmonic conjugates of P, Q, R w.r.t. B, C; C, A; A, B; prove that P′, Q′, R′ are collinear.

8. AA′ is a median of the triangle ABC and AD is drawn parallel to BC; prove that $A\{BC ; A'D\}$ is harmonic.

9. With the usual notation for the triangle ABC, prove that

$$A\{BC ; II_2\}, \quad A\{OH ; II_2\}, \quad D\{BFAE\}$$

are harmonic.

10. If I is the incentre of the triangle ABC, and if AI meets BC at Q and the circumcircle at P, prove that $PI^2=PA \,.\, PQ$.

11. The lines joining a point on a circle, centre O, to the extremities of a chord PQ meet the diameter perpendicular to PQ at A, B. Prove that $OA \,.\, OB=OP^2$.

12. The incircle of the triangle ABC touches BC, CA, AB at X, Y, Z; YZ produced meets BC at T; prove that $\{BXCT\}$ is harmonic.

13. ABCD is a parallelogram; AE is drawn parallel to BD; prove that $A\{EC ; BD\}$ is harmonic.

14. A′, B′, C′ are the mid-points of BC, CA, AB; prove that

$$A'\{AC ; B'C'\}$$

is harmonic.

15. AD, BE, CF are altitudes of the triangle ABC; DF meets BE at K and EF meets BC at M; prove that {BKHE} and {BDCM} are harmonic, where H is the orthocentre.

16. The tangent at a point C of a circle meets a diameter AB at T; prove that the other tangent from T to the circle is divided harmonically by CA, CB, T and its point of contact.

17. If $ax^2+2hx+b=0$, $a'x^2+2h'x+b'=0$ give two pairs of points on the x-axis which are harmonic conjugates, prove that $ab'+a'b=2hh'$.

18. A variable triangle PQR inscribed in a fixed circle has the side PQ parallel to a fixed chord and QR passes through the mid-point of the chord. Prove that RP passes through a fixed point.

19. ABCDPQ are six points on a circle; if P{ABCD} is harmonic, prove that Q{ABCD} is harmonic. What special case is obtained by taking Q very close to A?

20. (1) Two tangents TA, TB to a circle, centre C, are met by another tangent at P, Q; prove that $P\hat{C}Q$ is equal or supplementary to $\frac{1}{2}A\hat{C}B$.

(2) Hence show that if four tangents meet TA at P_1, P_2, P_3, P_4, and TB at Q_1, Q_2, Q_3, Q_4, and if $\{P_1P_2P_3P_4\}$ is harmonic, then $\{Q_1Q_2Q_3Q_4\}$ is harmonic.

Definition. *If four points on a circle are such that a harmonic pencil is formed by joining them to one other point on the circle, then Ex.* 19 *shows that a harmonic pencil is formed by joining them to any other point on the circle, and the four points are then said to form a* **harmonic** system **of** points *on the circle.*

If four tangents to a circle are such that they cut harmonically one other tangent to the circle, then Ex. 20 *shows that they cut harmonically every other tangent to the circle, and the four tangents are then said to form a* **harmonic** system **of tangents** *to the circle.*

21. (1) A, P, Q are three points on a circle, centre O; the tangents at P, Q meet the tangent at A at P_1, Q_1; prove that $P\hat{A}Q = P_1\hat{O}Q_1$ or $180° - P_1\hat{O}Q_1$.

(2) Hence prove that if P, Q, R, S form a harmonic system of points on a circle, the tangents at P, Q, R, S form a harmonic system of tangents to the circle.

22. PQ, AB are two parallel chords of a circle; C is the mid-point of AB; PC meets the circle at R; QT is the tangent at Q; prove that Q{ARBT} is harmonic.

23. AB is a diameter of a circle and AP any chord; the tangents at A, P meet at O; R is the foot of the perpendicular from P to AB; prove that BO bisects PR. [Prove that OP, OB, OR, OA cut AB harmonically.]

24. T is a point on the base BC produced of the triangle ABC ; N is the harmonic conjugate of T w.r.t. B, C ; TP is a tangent from T to the circle ABC ; PN meets AB, AC at L, M and the circle at Q ; prove that {PLQM} is harmonic.

25. ABC is a straight line ; AT is the tangent from A to the circle on BC as diameter ; O is its centre ; N is the foot of the perpendicular from T to BC ; prove that AO, AT, AN represent the Arithmetic, Geometric and Harmonic means between AB, AC ; and that these three are in geometrical progression.

THEOREM 36.

(1) If {ACBD}, {AC′B′D′} are harmonic, then CC′, BB′, DD′ are concurrent.

(2) If {ACBD}, {A′C′B′D′} are harmonic, and if AA′, CC′, BB′ are concurrent (at O say), then DD′ also passes through O.

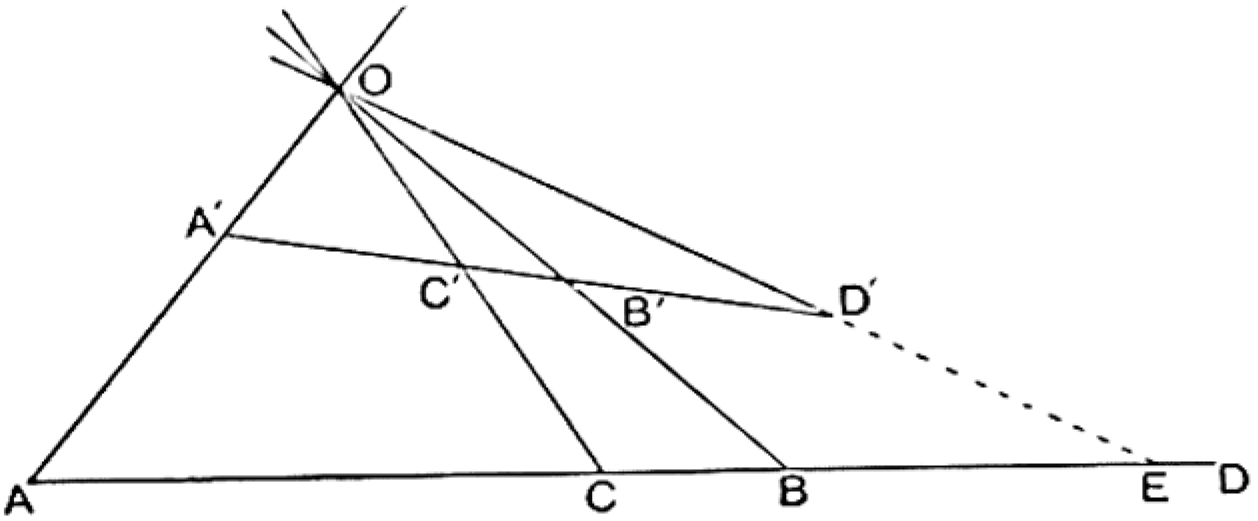

FIG. 45.

(1) is simply a special (*but important*) case of (2).

It is therefore only necessary to prove (2).

Join OD′ and produce it to meet AB at E.

Since O{A′C′B′D′} is harmonic, {ACBE} is harmonic ;

$$\therefore \frac{AC}{CB} = \frac{AE}{BE}, \quad \text{but} \quad \frac{AC}{CB} = \frac{AD}{BD};$$

$$\therefore \frac{AE}{BE} = \frac{AD}{BD};$$

$$\therefore \frac{AE}{AE - BE} = \frac{AD}{AD - BD} \quad \text{or} \quad \frac{AE}{AB} = \frac{AD}{AB};$$

∴ D coincides with E ;

∴ OD′ passes through D. Q.E.D.

THEOREM 37.

(1) **If O{ACBD}, O′{ACBD} are harmonic, and if O′ lies on OA, then C, B, D are collinear.**

(2) **If O{ACBD}, O′{ACBD} are harmonic, and if A, C, B are collinear, then D also lies on ACB.**

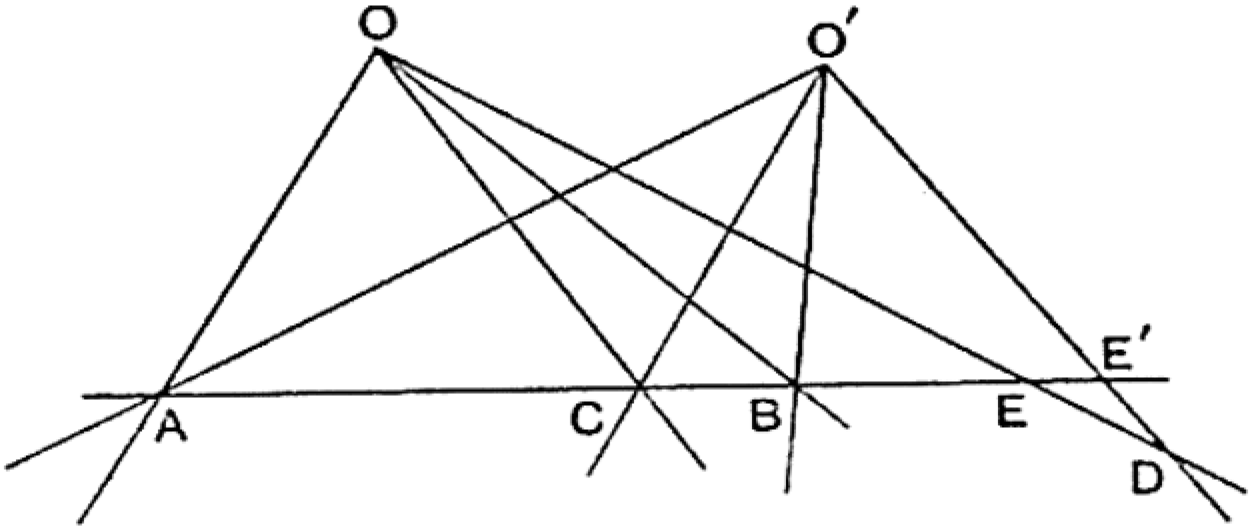

FIG. 46.

(1) is simply a special (*but important*) case of (2).

It is therefore only necessary to prove (2).

Produce ACB to meet OD, O′D at E, E′.

Since O{ACBD} is harmonic, {ACBE} is harmonic;

$$\therefore \frac{AC}{CB}=\frac{AE}{BE}.$$

Similarly, $\frac{AC}{CB}=\frac{AE'}{BE'}$, since {ACBE′} is harmonic;

$$\therefore \frac{AE}{BE}=\frac{AE'}{BE'};$$

∴ as in Theorem 36, E coincides with E′.

∴ ACB meets OD, O′D at the same point, *i.e.* at their point of intersection D.

∴ A, C, B, D are collinear. Q.E.D.

26. If {APQR}, {AP′Q′R′} are harmonic, prove that PR′, P′R, QQ′ are concurrent.

27. If, in the triangle ABC, AI, I_2I_3 cut BC at L, M and CI, I_1I_2 cut AB at P, Q, prove that PL, AC, QM are concurrent and that PM, QL intersect on AC.

28. If ACBD form a harmonic system of points on a circle, prove that the following pairs of lines intersect at collinear points: AC, BD; AB, CD; the tangents at A, D; the tangents at B, C.

29. Through a fixed point A is drawn a variable line meeting two fixed lines OC, OD at P_1, P_2; a point Q is taken on AP_1P_2 such that

$$\frac{1}{AQ}=\frac{1}{AP_1}+\frac{1}{AP_2};$$

prove that the locus of Q is a straight line.

30. AD, BE, CF are altitudes of the triangle ABC; DE, EF meet AB, BC at F′, D′; prove that FD, F′D′ intersect on AC.

31. If, in the triangle ABC, AO, II_1, I_2I_3 meet BC, DE, DF at P, Q, R, prove that P, Q, R are collinear.

32. With the usual notation for the triangle ABC, if C′D meets AC at K, EF meets BC at P, prove that KP is parallel to AB.

33. With the notation of Ex. 32, if C′P meets AC at Q, prove that DQ is parallel to AB.

34. The tangent at a point P of a circle cuts a diameter AB at T; PN is the perpendicular to AB; the line joining B to the mid-point of TP cuts PN at Q; prove that AQ is parallel to TP.

CROSS RATIO.

If a transversal meets four lines which pass through a point O at A, C, B, D, the proof of Theorem 34, shows that

$$\frac{AC \cdot BD}{AD \cdot BC}=\frac{\sin AOC \cdot \sin BOD}{\sin AOD \cdot \sin BOC};$$

and consequently, if A′C′B′D′ is any other transversal,

$$\frac{A'C' \cdot B'D'}{A'D' \cdot B'C'}=\frac{AC \cdot BD}{AD \cdot BC}.$$

If {ACBD} is harmonic, each of these expressions $= -1$.

Consequently the fact that {ACBD} is harmonic is sometimes denoted by $\{ACBD\} = -1$.

Definitions.

(1) Any set of collinear points is said to form a *range*.

(2) Any set of concurrent lines is said to form a *pencil*: the lines are called *rays* of the pencil and the point of concurrency is called its *vertex*.

(3) If a transversal meets a pencil O{ACBD} at A, C, B, D, the ratio $\frac{AC \cdot BD}{AD \cdot BC}$ or $\frac{\sin AOC \cdot \sin BOD}{\sin AOD \cdot \sin BOC}$ is called the *cross-ratio* of the pencil O{ACBD} or the *cross-ratio* of the range {ACBD}.

(4) If two ranges or pencils are of equal cross-ratio, they are said to be *equi-cross.*

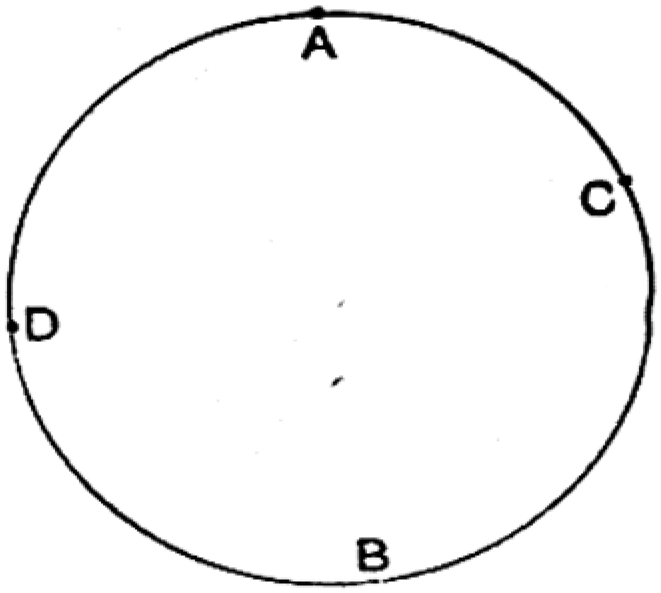

FIG. 47.

To remember the cross-ratio expression for {ACBD}, place the four points in order on a circle: the numerator is obtained by reading forwards from A and the denominator by reading backwards from A.

THEOREM 38.

If {ACBD} = {ADBC}, then {ACBD} is harmonic.

It is given that $$\frac{AC \cdot BD}{AD \cdot BC} = \frac{AD \cdot BC}{AC \cdot BD};$$

$$\therefore \frac{AC^2}{CB^2} = \frac{AD^2}{DB^2} \quad \text{or} \quad \frac{AC}{CB} = \pm\frac{AD}{DB}.$$

Now, if $\frac{AC}{CB} = \frac{AD}{DB}$, C coincides with D from the proof of Theorem 36.

$$\therefore \frac{AC}{CB} = -\frac{AD}{DB};$$

∴ {ACBD} is harmonic. Q.E.D.

It follows at once that if O{ACBD} = O{ADBC}, then O{ACBD} is harmonic.

THEOREM 39.

(1) If {ACBD}, {AC′B′D′} are equi-cross, then CC′, BB′, DD′ are concurrent.

(2) If {ACBD}, {A′C′B′D′} are equi-cross, and if AA′, BB′, CC′ are concurrent (at O say), then DD′ also passes through O.

(3) If O{ACBD}, O′{ACBD} are equi-cross, and if O′ lies on OA, then C, B, D are collinear.

(4) If O{ACBD}, O′{ACBD} are equi-cross, and if A, C, B are collinear, then D also lies on ACB.

The method of proof is that of Theorems 36, 37.

35. **Prove that the value of the cross-ratio {ACBD} is unaltered, if, when any two letters are interchanged, the other two letters are also interchanged.** (*e.g.* {ACBD} = {CADB}).

36. **If** {ACBD} $= l$, **prove that** {ABCD} $= 1 - l$;

$$\{ABDC\} = \frac{l-1}{l}; \qquad \{ACDB\} = \frac{l}{l-1};$$

$$\{ADBC\} = \frac{1}{l}; \qquad \{ADCB\} = \frac{1}{1-l}.$$

37. **Prove that there are 24 cross-ratios of four collinear points, and that they fall into sets of four, each of which is equal to one of the six ratios in Ex. 36.**

38. Prove that the six different cross-ratios of four collinear points can be written as l, m, n, $\frac{1}{l}$, $\frac{1}{m}$, $\frac{1}{n}$, where

$$l + \frac{1}{m} = m + \frac{1}{n} = n + \frac{1}{l} = -lmn = 1.$$

39. Prove that $\frac{AB}{CB}$ = {ABC∞}, where ∞ represents the point at infinity on AC.

40. A variable line meets four fixed planes which have a common line of intersection at P, Q, R, S; prove that {PQRS} is constant.

41. ABCD is a transversal of O{ABCD}; through B is drawn a line FBG parallel to OD to meet OA, OC at F, G; prove that

$$\{ABCD\} = \frac{FB}{GB}.$$

42. PQR is a variable triangle such that each side passes through a fixed point; if the three fixed points are collinear, and if P, Q move on fixed lines, prove that the locus of R is a fixed line. [Take three positions of the triangle, and prove that the three positions of R are collinear, using Th. 39 (3).]

43. PQR is a variable triangle with its vertices on three fixed concurrent lines; PQ, QR pass through fixed points; prove that RP also passes through a fixed point.

44. (**Desargues' theorem on perspective triangles.**) Two triangles ABC, A′B′C′ are such that AA′, BB′. CC′ are concurrent, at O say; if BC, B′C′ meet at X; CA, C′A′ at Y; AB, A′B′ at Z; prove that X, Y, Z are collinear. [Produce CB, C′B′ to cut OA at E, E′ and prove A{XYZO} = A′{XYZO}.]

45. (**Desargues' theorem on the quadrangle.**) Six lines are drawn joining in pairs the points A, B, C, D. Any other straight line meets AB, CD, AD, BC, AC, BD at P, P′, Q, Q′, R, R′; prove that

$$\{PQRR'\} = \{P'Q'R'R\}.$$

[Project the ranges from A, C on to BD.]

46. A, B, C and A′, B′, C′ are two sets of collinear points; the pairs of lines AC, A′C′; AC′, B′C; A′C, AB′; A′B, AB′; A′C, AC′; B′C, BC′ intersect at O, K, H, P, Q, R. Prove that

(1) {APHB′} = {ABCO} = {KRCB′}; (2) P, Q, R are collinear.

47. ABC is a given triangle and D a fixed point on its circumcircle; a variable line through D cuts AB, BC, CA at C′, A′, B′ and the circumcircle at P; prove that {PA′B′C′} is of constant cross-ratio.

CHAPTER VII.

THE QUADRILATERAL AND QUADRANGLE.

THIS chapter will serve as an introduction to what is known as the *Principle of Duality.*

Definitions.

(1) If A and B are two points, the unlimited line joining A and B is called the *join* of A and B.

(2) If a and b are two lines, the point of intersection of a and b is called the *meet* of a and b.

There are two distinct points of view from which any curve may be regarded: (1) as a *locus*; (2) as an *envelope.*

I. A circle, for example, may be considered as formed by a series of points each of which is at a constant distance from a fixed point; in order to pass from the series of points to the continuous curve, it is necessary to imagine that a number of lines are drawn joining consecutive points of the series: by taking a large enough number of these points, each connecting line can be made as short as we please: and the limiting result of the process is said to constitute the curve.

From this point of view, the curve is regarded as a *locus* or the path of the moving (tracing) point.

II. A circle may, however, also be considered as generated by a series of straight lines, each of which is at a constant distance from a fixed point. On each line is cut off a segment by the two lines which come, one before and the other after it, in the series: by taking a large enough number of these lines, each intercepted segment can be made as short as we please: and the limiting result of the process is said to constitute the curve.

From this point of view, the curve is regarded as an *envelope*, or a curve generated by a moving (tangent) line.

The essential difference between these two view-points is that in I. the *point* is the fundamental conception and the straight line arises as the join of two points, whereas in II. the *straight line* is the fundamental conception, and the point arises as the meet of two lines.

If then we have any geometrical system built up on one of these conceptions, it is possible to construct the corresponding system which is based on the other. Theorems affecting the disposition of points and lines therefore occur in pairs, so that the knowledge of one includes the knowledge of the other. This correspondence is called *The Principle of Duality*.

In what follows, it is convenient to take single capital letters to represent points and single small letters to represent lines. Correspondence will be represented in parallel columns.

Definitions.

AB is the *join* of two points A, B. $\{ABCD\}$ is the range formed by four collinear points A, B, C, D.	ab is the *meet* of two lines a, b. $\{abcd\}$ is the pencil formed by four concurrent lines a, b, c, d.
$H\{ABCD\}$ is the pencil formed by the joins of the *vertex* H to the four points A, B, C, D.	$h\{abcd\}$ is the range formed by the meet of the *base* h with the four lines a, b, c, d.
A figure formed by three points and their joins is called a *triangle*.	A figure formed by three lines and their meets is called a *trilateral*.
A curve which is met by any straight line at two, and only two, points is called a *curve of the second degree*.	A curve to which two, and only two, tangents can be drawn from any point is called a *curve of the second class*.

Or,

The locus of a point which moves so that in two, and only two, of its positions it lies on any straight line is called a *curve of the second degree*.	The envelope of a line which moves so that in two, and only two, of its positions it passes through any point is called a *curve of the second class*.

Theorems.

The joins of n points, no three of which are collinear, taken in pairs, form $\frac{1}{2}n(n-1)$ lines.	The meets of n lines, no three of which are concurrent, taken in pairs, form $\frac{1}{2}n(n-1)$ points.
A, B, C and A', B', C' are the vertices of two triangles; if the joins of A, A'; B, B'; C, C' are concurrent, then the meets of BC, $B'C'$; CA, $C'A'$; AB, $A'B'$ are collinear.	a, b, c and a', b', c' are the sides of two trilaterals; if the meets of a, a'; b, b'; c, c' are collinear, then the joins of bc, $b'c'$; ca, $c'a'$; ab, $a'b'$ are concurrent.
If the sides BC, CA, AB of a variable triangle ABC pass through three fixed collinear points, and if A, B lie on two fixed lines, then C lies on a fixed line.	If the vertices bc, ca, ab of a variable trilateral abc lie on three fixed concurrent lines, and if a, b pass through two fixed points, then c passes through a fixed point.
If {ABCD}, {$AB'C'D'$} are two harmonic ranges on different bases, then the joins of B, B'; C, C'; D, D' are concurrent.	If $\{abcd\}$, $\{ab'c'd'\}$ are two harmonic pencils with different vertices, then the meets of b, b'; c, c'; d, d' are collinear.

A complete quadrangle is formed by four points A, B, C, D, called its *vertices*, no three of which are collinear. Their joins form six straight lines, grouped in pairs, AB, CD ; BC, AD ; CA, BD, called its *sides* ; the sides in each pair being called opposite sides.

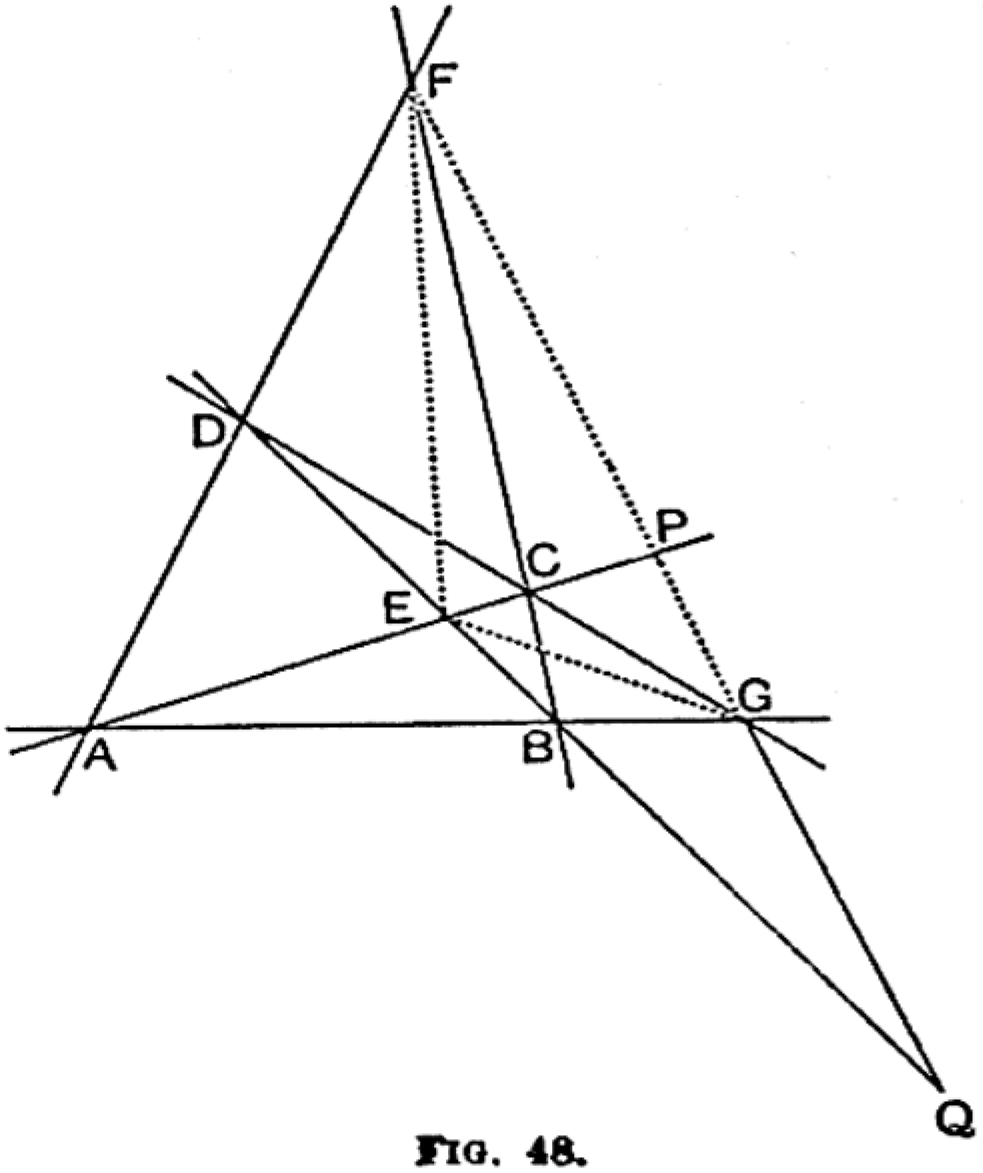

FIG. 48.

The meets of opposite sides form three points G, F, E, called *diagonal points* ; and the triangle EFG is called the *diagonal point triangle*.

Now, {AECP} = F{AECP} = {DEBQ} = G{DEBQ} = {CEAP}.

∴ by Theorem 38, {AECP} is harmonic.

It follows at once that each set of 4 collinear points in Fig. 48 is harmonic.

This is known as the harmonic property of the complete quadrangle

A more elementary proof is given in Theorem 40.

A complete quadrilateral is formed by four lines a, b, c, d, called its *sides*, no three of which are concurrent, Their meets form six points, grouped in pairs, ab, cd; bc, ad; ca, bd, called its *vertices*; the vertices in each pair being called opposite vertices.

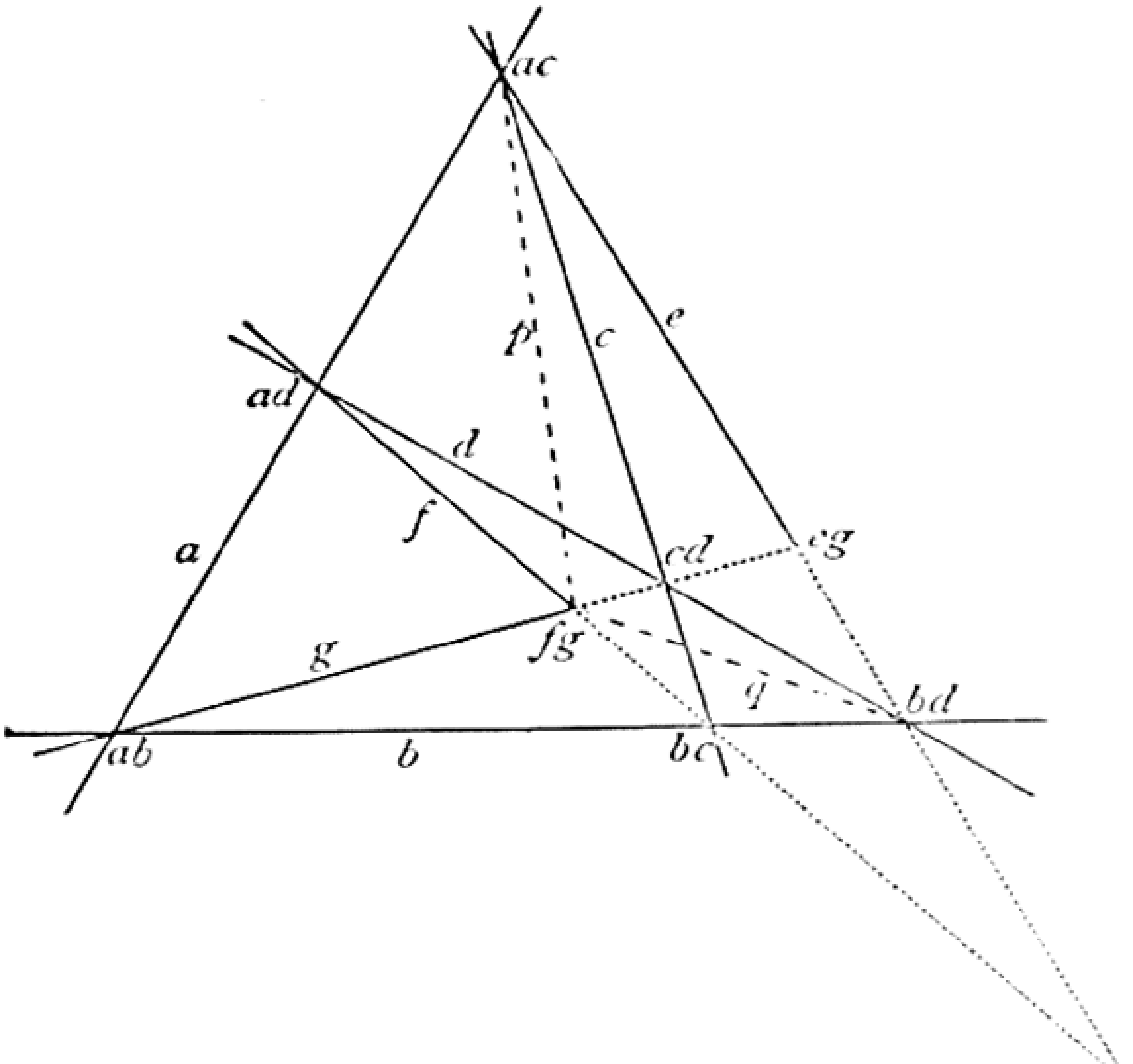

FIG. 49.

The joins of opposite vertices form three lines g, f, e, called *diagonal lines*; and the triangle efg is called the *diagonal line trilateral.*

Now, $\{aecp\} = f\{aecp\} = \{debq\} = g\{debq\} = \{ceap\}$.

$\therefore$ by Theorem 38, $\{aecp\}$ is harmonic.

It follows at once that each set of 4 concurrent lines in Fig. 49 is harmonic.

This is known as the harmonic property of the complete quadrilateral.

THEOREM 40.

ABCD is a quadrangle: AD, BC meet at F; AB, CD meet at G; AC, BD meet at E; FE meets AB at K; then {AKBG} is harmonic.

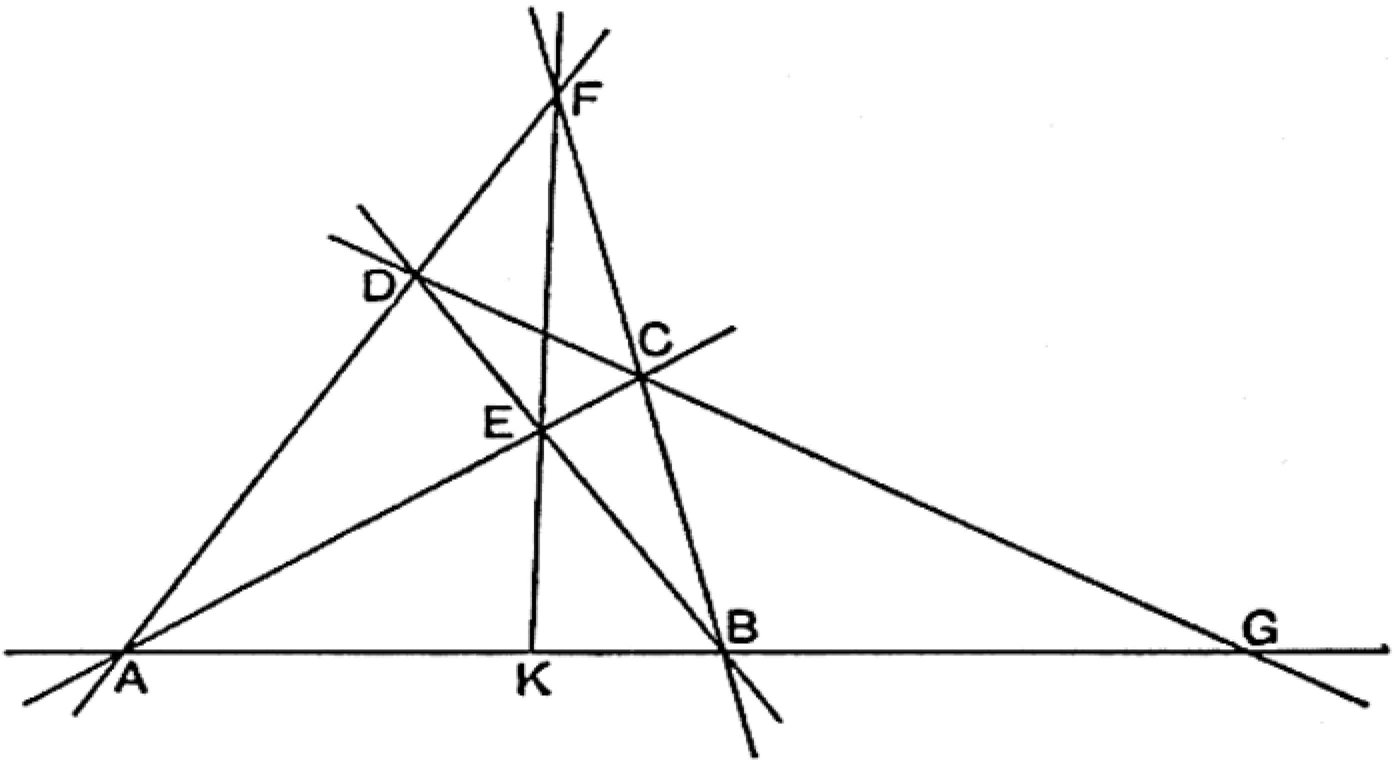

FIG. 50.

From the triangle FAB,

$$\frac{\mathrm{AK}}{\mathrm{KB}}\cdot\frac{\mathrm{BC}}{\mathrm{CF}}\cdot\frac{\mathrm{FD}}{\mathrm{DA}}=1,\quad \text{by Ceva's theorem,}$$

and

$$\frac{\mathrm{AG}}{\mathrm{GB}}\cdot\frac{\mathrm{BC}}{\mathrm{CF}}\cdot\frac{\mathrm{FD}}{\mathrm{DA}}=-1,\quad \text{by Menelaus' theorem.}$$

$$\therefore\ \frac{\mathrm{AK}}{\mathrm{KB}}=-\frac{\mathrm{AG}}{\mathrm{GB}};$$

$\therefore$ {AKBG} is harmonic. Q.E.D.

It is easy to deduce from this theorem that every set of four collinear points, and of four concurrent lines, in Fig. 51 is harmonic. This is left, as an exercise, to the student.

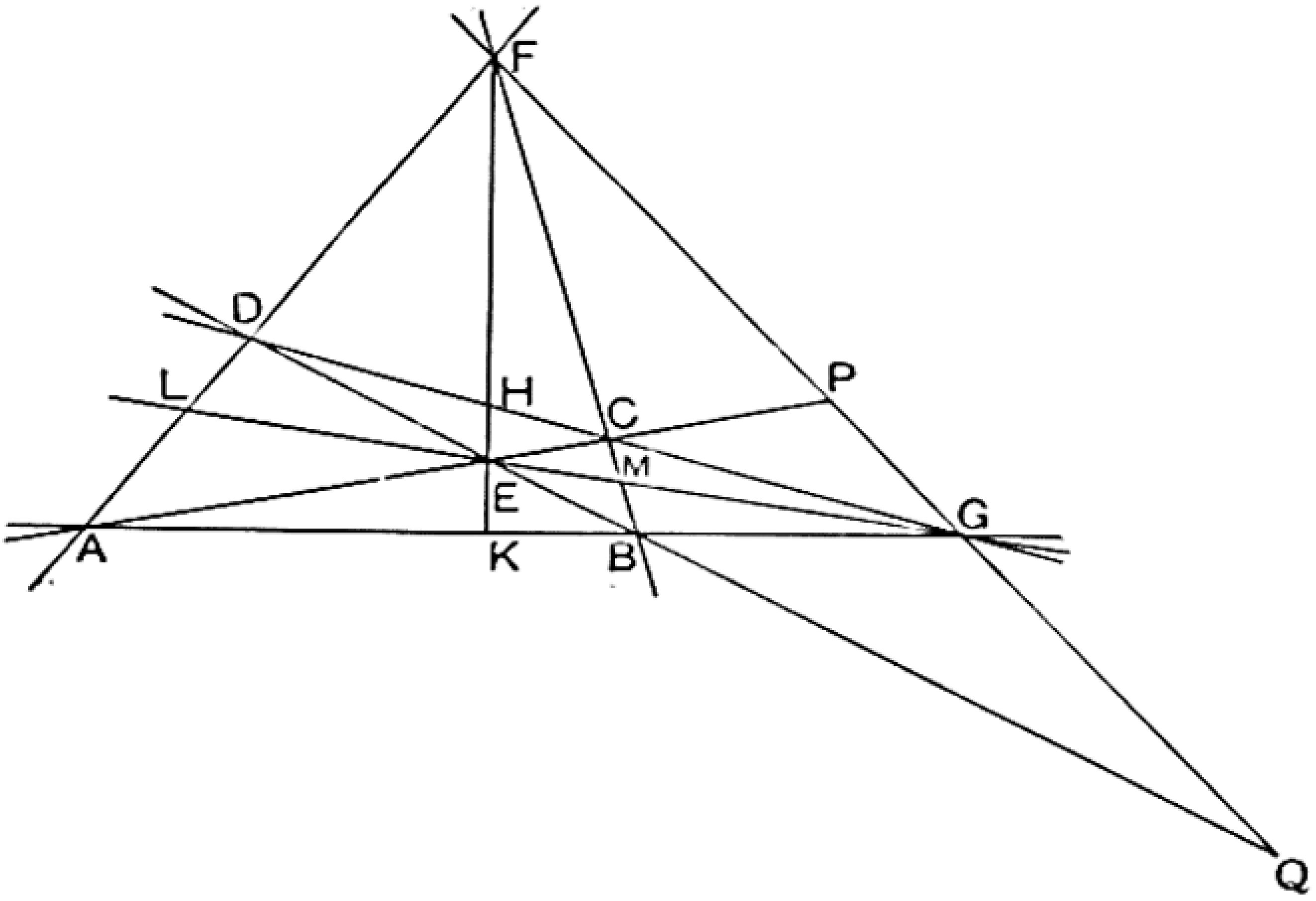

FIG. 51.

Write down, *without proof*, the dual theorems of Ex. 1-9; draw and letter figures for each pair of theorems. [Compare Figs. 48, 49.]

1. From a fixed point D, two variable lines DPQ, DRS are drawn to cut two fixed lines AB, AC at P, Q and R, S; then the locus of the meet of PS, QR is a straight line.

2. {ABCD}, {AB′C′D′} are two harmonic ranges on different bases, then BD′, B′D, CC′ are concurrent.

3. {ABCD}, {A′B′C′D′} are two harmonic ranges, such that AA′ BB′, CC′ concur at O, then DD′ passes through O.

4. Two lines FDA, FCB are met by two other lines GBA, GCD; BD meets AC at E; then F{AB; EG} is harmonic.

5. A_1, A_2, ... A_n are n coplanar points which lie on fixed concurrent lines; if A_1A_2, A_2A_3, ... $A_{n-1}A_n$ pass through fixed points, then A_rA_s passes through a fixed point.

6. One, and only one, curve of the second degree can be drawn to pass through five given points, no three of which are collinear.

7. $A_1A_2A_3A_4A_5A_6$ are six points on a curve of the second degree, then the meets of A_1A_2, A_4A_5; A_2A_3, A_5A_6; A_3A_4, A_6A_1 are collinear.

8. A variable curve of the second degree passes through four fixed points A, B, C, D; if two fixed lines AX, BY cut the curve at P, Q, then PQ passes through a fixed point.

9. The sides of a variable triangle pass through three non-collinear fixed points; if two of the vertices lie on fixed lines, then the locus of the third vertex is a curve of the second degree.

---

10. Given three collinear points A, B, C, by using a ruler only, show how to construct a point D such that {ABCD} is harmonic.

11. Given three concurrent lines PA, PB, PC, by using a ruler only, show how to construct a line PD such that P{ABCD} is harmonic.

12. K is the given mid-point of the line joining two given points A, B; O is a given point outside AB; draw through O a line parallel to AB, using a ruler only.

13. Given three points A, B, C on a given circle, with the use of a ruler construct a point D such that A, B, C, D form a harmonic system of points on the circle.

14. The straight lines AD, BC cut at C; on BC any point E is taken; AE meets BD at Q; CQ meets ED at R; BR cuts AD at P. Prove that P remains stationary, while E moves along BC.

15. With the notation of Fig. 48, prove that (1) PD, GE, AQ and (2) QC, EG, PB are concurrent.

16. With the notation of Fig. 51, prove that (1) L, H, P; (2) K, M, P are collinear.

17. C is a point on the line of intersection AB of two planes; points X, X′ are taken, one on each plane; a line CHK meets AX, BX at H, K; a line CH′K′ meets AX′, BX′ at H′, K′; AK, AK′ meet BH, BH′ at Y, Y′ respectively; prove that X, X′, Y, Y′ are coplanar.

18. ABCD is a quadrilateral, having AB parallel to CD; AC, BD meet at P; AD, BC meet at Q. Prove that PQ bisects AB.

19. The base BC of a triangle is given, and the vertex P moves on a fixed line: another fixed line cuts PB, PC at H, K; find the locus of the meet of CH, BK.

20. Given two lines p, q which meet off the paper, and a point A, construct, with the use only of a ruler, the join of A to the meet of p, q

* THEOREM 41.

The mid-points of the three diagonals of a complete quadrilateral are collinear.

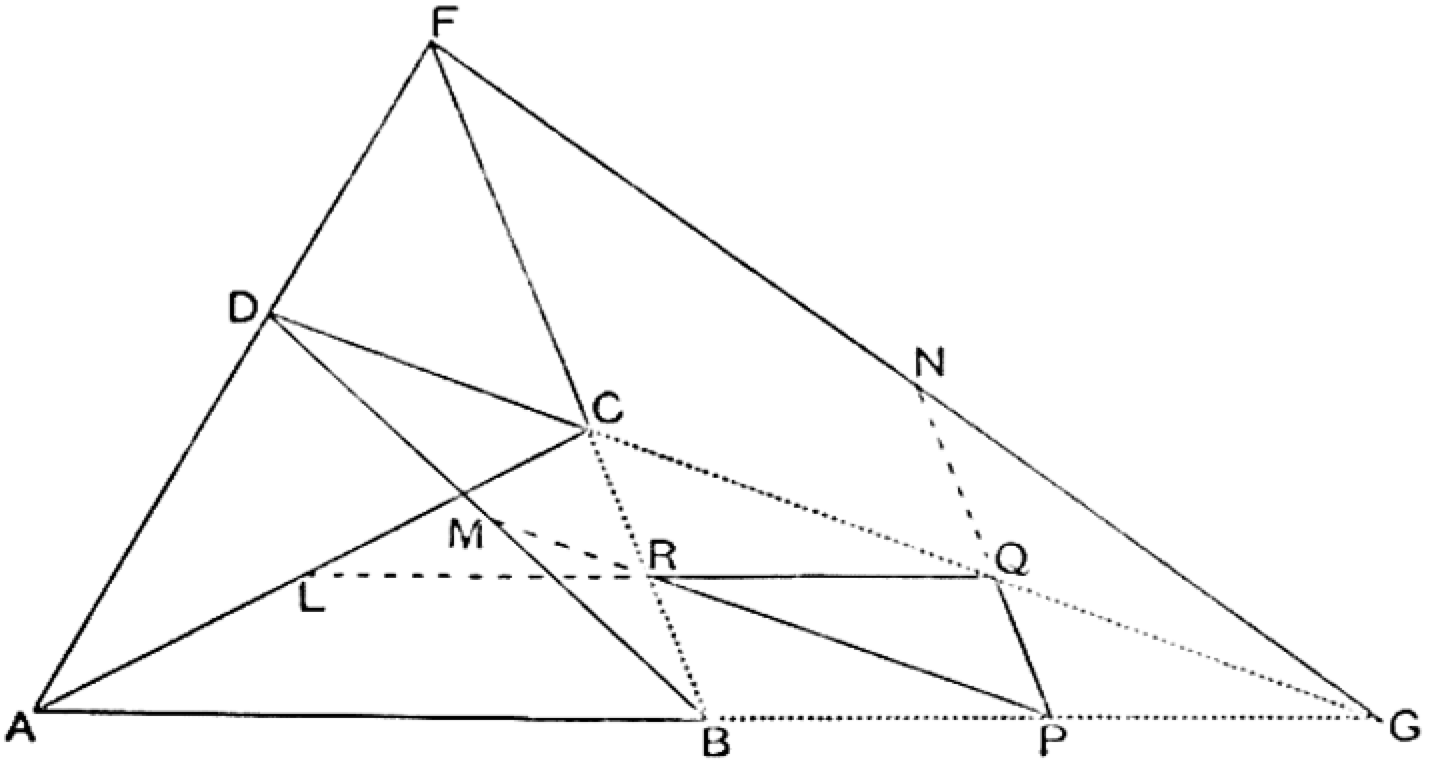

FIG. 52.

AC, BD, FG are the diagonals of the quadrilateral ABCD.

Let P, Q, R be the mid-points of BG, GC, CB; and produce PQ, QR, RP to meet FG, AC, BD at N, L, M.

PQ, QR, RP are respectively parallel to CB, BG, GC.

Since PQN is parallel to BF, and since GP = PB, ∴ GN = NF.

∴ N is the mid-point of FG, and similarly L, M are the mid-points of AC, BD.

Now, by parallels,

$$\frac{RL}{LQ} = \frac{BA}{AG}; \quad \frac{QN}{NP} = \frac{CF}{FB}; \quad \frac{PM}{MR} = \frac{GD}{DC}.$$

$$\therefore \frac{RL}{LQ} \cdot \frac{QN}{NP} \cdot \frac{PM}{MR} = \frac{BA}{AG} \cdot \frac{GD}{DC} \cdot \frac{CF}{FB} = -1,$$

by Menelaus, since ADF is a transversal of the triangle BCG.

∴ L, M, N are collinear, converse of Menelaus. Q.E.D.

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* This proof was given by Mr. C. E. Hillyer in the *Educational Times*

Theorem 41 also admits of a statical proof.

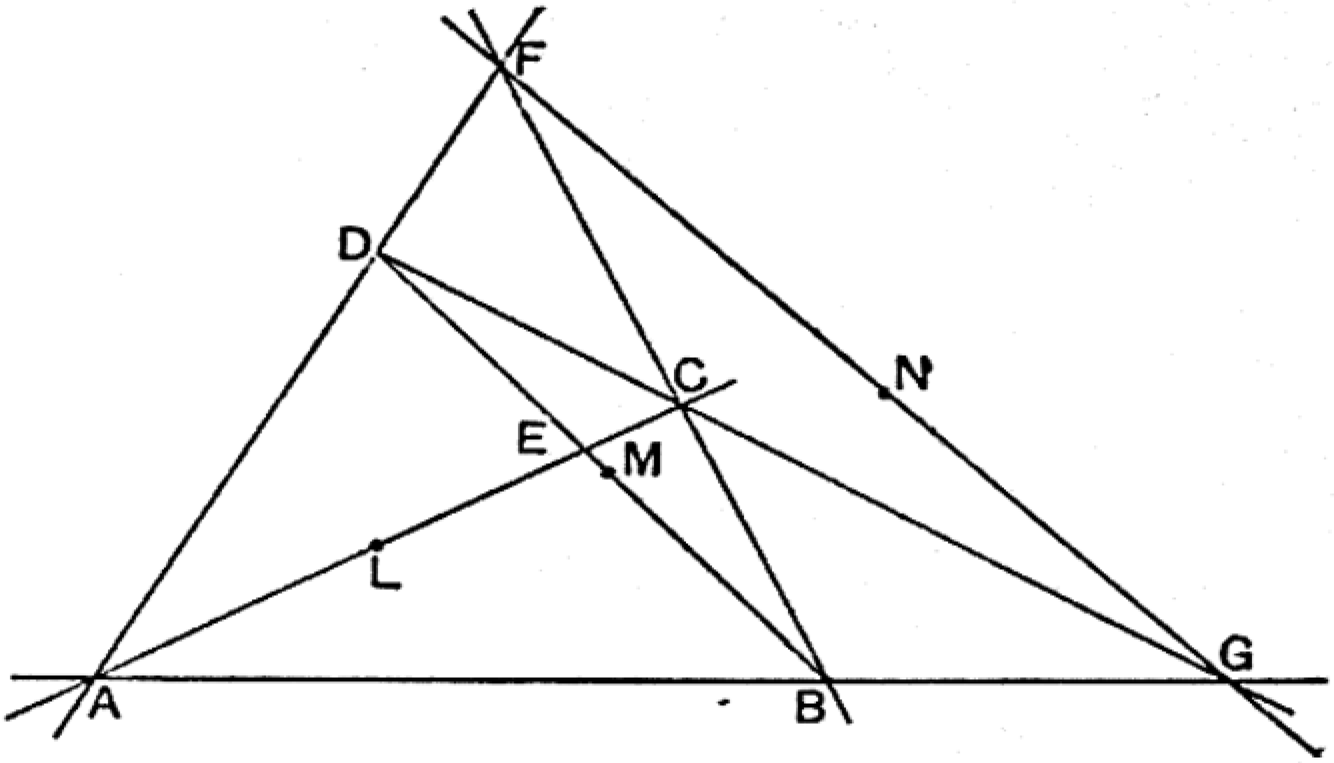

FIG. 53.

L, M, N are the mid-points of AC, BD, FG.

Consider the system of forces $\overline{AD}$, $\overline{AB}$, $\overline{CD}$, $\overline{CB}$.

$$\overline{AD}+\overline{AB}=2\overline{AM} \quad \text{and} \quad \overline{CD}+\overline{CB}=2\overline{CM}.$$

∴ the resultant of the system passes through M, and similarly through L. [In fact, it is $4\overline{LM}$.]

Now the system is also equivalent to

$$\begin{aligned} &\overline{AF}+\overline{FD}+\overline{AG}+\overline{GB}+\overline{CG}+\overline{GD}+\overline{CF}+\overline{FB} \\ &=\overline{AF}+\overline{AG}+\overline{FD}+\overline{GD}+\overline{FB}+\overline{GB}+\overline{CG}+\overline{CF} \\ &=2\overline{AN}+2\overline{ND}+2\overline{NB}+2\overline{CN}. \end{aligned}$$

∴ the resultant passes through N.

∴ L, M, N are collinear. Q.E.D.

Further, if a circle can be inscribed in the quadrilateral, its centre must lie on LMN.

For suppose the circle touches AB, BC, CD, DA at P, Q, R, S; then the force system is equivalent to

$$(\overline{AP}+\overline{AS})+(\overline{PB}+\overline{QB})+(\overline{CR}+\overline{CQ})+(\overline{RD}+\overline{SD}).$$

Now the resultant of each pair bisects the corresponding chord of contact and therefore passes through the centre of the circle, so that the resultant must pass through the centre of the circle.

Q.E.D.

This method yields an immediate proof of the theorem that " the locus of the centres of conics touching four straight lines is a straight line."

Use statical methods for Ex. 21-24.

21. Through a point F inside a rectangle ABCD, lines are drawn parallel to the sides meeting AB, BC, CD, DA at G, E, K, H; prove that BH, CF, DG are concurrent.

22. ABCD is a quadrilateral; if AB = CD, prove that the line joining the mid-points of BC and AD is equally inclined to the other two sides.

23. A parallelogram ABCD is divided into four parallelograms by two lines PKQ, RKS meeting DC, AB, DA, CB at P, Q, R, S. Prove that the triangle AKC is equal to half the difference of the parallelograms KB, DK.

24. Prove that the three pairs of bisectors of the opposite angles of a complete quadrilateral intersect at three collinear points.

25. With the notation of Fig. 53, if ABCD is inscribed in a circle, prove that $FG^2 = FR^2 + GS^2$, where FR, GS are the tangents from F, G to the circle.

26. If two of the opposite angles of a complete quadrilateral are right angles, prove that the third diagonal is perpendicular to one of the internal diagonals.

CHAPTER VIII.

ORTHOGONAL CIRCLES.

Definitions.

(1) If two curves intersect at a point A, and if the tangents at A to the curves make an angle α with each other, the two curves are said to cut at A at an angle α.

In particular, if $\alpha = 90°$, the curves are said to cut *orthogonally* or at right angles: and the curves are called *orthogonal curves.*

(2) If CA is a radius of a circle, centre C, and if P, Q are two points on CA, such that $CP . CQ = CA^2$, then P, Q are called ***inverse points*** w.r.t. the circle.

THEOREM 42.

If two circles, centres A and B, intersect orthogonally at C, then (1) CA, CB are tangents to the circles and (2) $CA^2 + CB^2 = AB^2$.

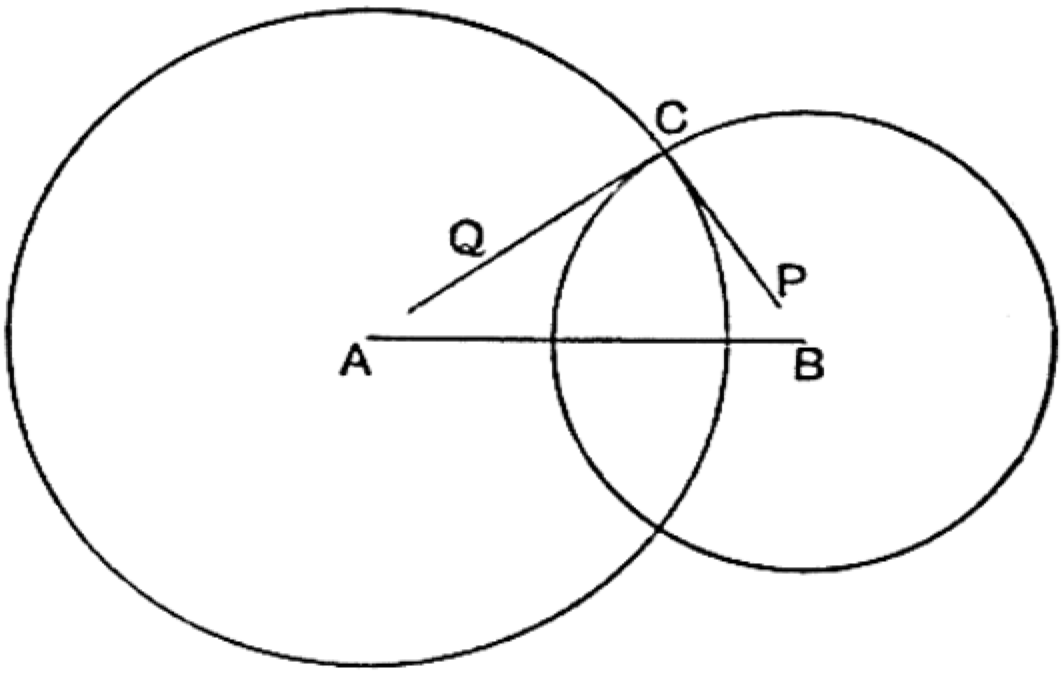

FIG. 54.

(1) Draw the tangents CP, CQ to the circles.

Since $Q\hat{C}P = 90^\circ$, CQ is perpendicular to the tangent CP to the circle, centre A, and therefore passes through A.

Similarly CP passes through B.

$\therefore$ AQC, BPC are tangents to the circles. Q.E.D.

(2) Since, from (1), $A\hat{C}B = 90^\circ$,

$$AB^2 = AC^2 + CB^2. \qquad \text{Q.E.D.}$$

The converse theorems, which are easily proved, are most important.

If two circles, centres A, B, intersect at C, and if *either* $A\hat{C}B = 90^\circ$, *or* $AB^2 = AC^2 + CB^2$, *or* if AC is a tangent to the circle, centre B, then the two circles are orthogonal.

THEOREM 43.

Two circles intersect at C, D; if the sum of the angles in the segments of the two circles on opposite sides of CD is 90° or 270°, then the circles cut orthogonally.

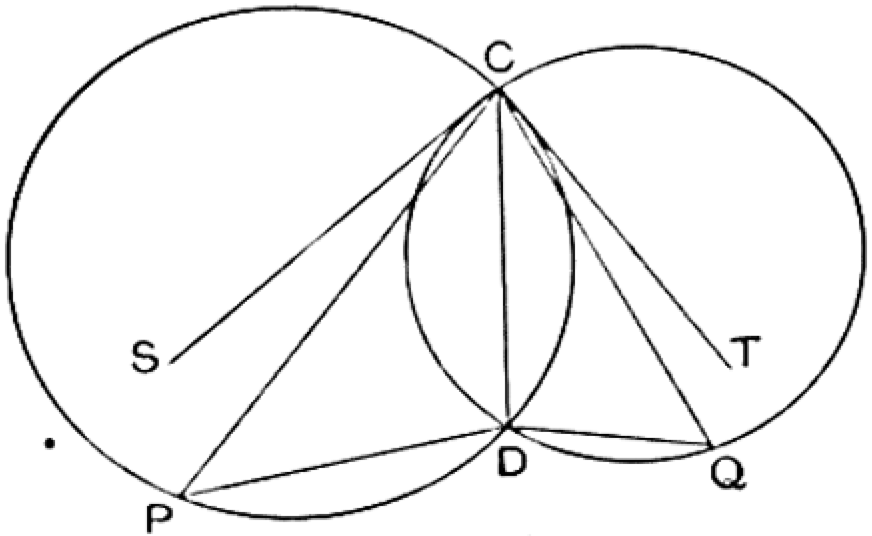

FIG. 55.

Draw the tangents CS, CT.

$$C\hat{P}D = D\hat{C}T \quad \text{and} \quad C\hat{Q}D = S\hat{C}D, \text{ alternate segment.}$$

$$\therefore \; S\hat{C}T = C\hat{P}D + C\hat{Q}D = 90^\circ.$$

$\therefore$ the circles cut orthogonally. Q.E.D.

If P, Q are taken on the minor arcs CD, we should have the case $C\hat{P}D + C\hat{Q}D = 270^\circ$.

1. If two circles intersect at A and B, prove that the angles at which they cut each other at A and B are equal.

2. If P, Q are inverse points w.r.t. a circle, and if PQ meets the circle at A, B, prove that {AB, PQ} is harmonic.

3. P is a point on a circle, centre C; AB is a diameter: prove that the circles APC, BPC are orthogonal.

4. A common tangent to two circles touches them at P, Q; prove that the circle on PQ as diameter is orthogonal to each of them.

5. {ACBD} is harmonic; prove that the circles on AB, CD as diameters are orthogonal.

6. Two circles, centres A, B, intersect at C; prove that the angle at which they cut equals $180° - A\hat{C}B$.

7. If AHD, BHE, CHF are the altitudes of the triangle ABC, prove that the three circles whose centres are A, B, C and the squares of whose radii are AH . AD, BH . BE, CH . CF cut orthogonally.

8. A line QPR is drawn through the meet P of two circles and cuts them again at Q, R; with Q, R as centres, two circles are described orthogonal to the given circles; prove that they intersect on the circle whose diameter is QR.

9. Prove that the circle whose diameter is the third diagonal of a quadrilateral inscribed in a circle, is orthogonal to that circle.

10. If H is the orthocentre of the triangle ABC, prove that the circles on AH and BC as diameters are orthogonal.

11. A point C outside a circle is joined to the ends A, B of a diameter and the joins cut the circle at P, Q; prove that the circle CPQ is orthogonal to the given circle.

12. TP, TQ are the tangents from T to a circle and R is any point such that TR = TP; RP, RQ meet the circle again at A, B: prove that AB is a diameter.

13. With the usual notation, prove that the circle IBC is orthogonal to the circle on I_2I_3 as diameter

THEOREM 44.

(1) If {ACBD} is harmonic, then the circle on AB as diameter is orthogonal to any circle through C and D.

(2) If C, D are inverse points to a circle, any circle through C, D is orthogonal to that circle.

(3) If two circles cut orthogonally, then any diameter of one is cut harmonically by the other.

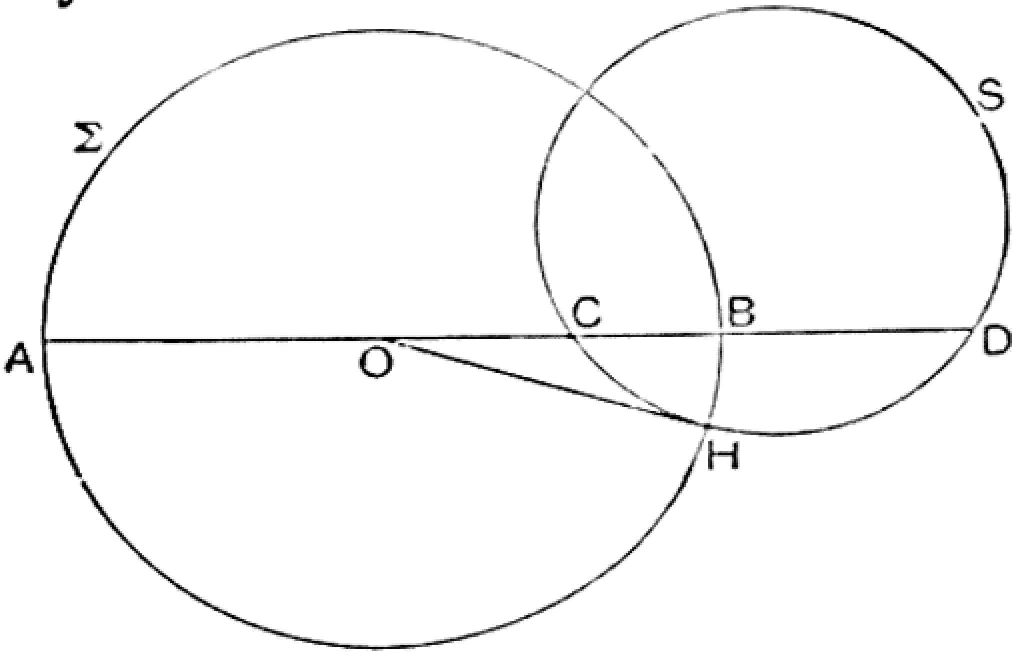

FIG. 56.

(1) Let O be the mid-point of AB, and H a meet of the circles.

Since {ACBD} is harmonic and AO = OB,

$$\therefore\ OC \cdot OD = OB^2 = OH^2.$$

∴ OH is a tangent to the circle CHD.

∴ the circles cut orthogonally. Q.E.D.

(2) If C, D are inverse points,

$$OC \cdot OD = OB^2 = OH^2.$$

∴ as in (1), the circles are orthogonal. Q.E.D.

(3) Let Σ and S be the circles, and let any diameter AB of Σ meet S at C, D. Let O be the centre of Σ.

Since the circles are orthogonal, OH is a tangent.

$$\therefore\ OC \cdot OD = OH^2 = OA^2 = OB^2.$$

∴ {ACBD} is harmonic. Q.E.D.

It follows from the proof of (3) that C, D are inverse points w.r.t. Σ. Consequently, if two circles are orthogonal, any diameter of one is cut by the other at inverse points.

14. A fixed diameter AB of a circle S meets a fixed orthogonal circle at C, D; P is a variable point on S; prove that $\frac{PC}{PD}$ is constant.

15. Prove that the circumcircle of the triangle formed by the three diagonals of a complete quadrilateral is orthogonal to each of the three circles whose diameters are the three diagonals.

16. Two circles S_1, S_2 intersect at P, Q; another circle Σ cuts S_1 and S_2 orthogonally; prove that P, Q are inverse points w.r.t. Σ.

17. Construct a circle to pass through two given points and to be orthogonal to a given circle.

18. Two circles, centres A, B, cut orthogonally at Q, Q′; a line PQR cuts the circles again at P, R; if L is the mid-point of PR, prove that $A\hat{L}B = 90°$. (Use LQ . LR = PL . LQ.)

19. Prove that a line, cut harmonically by two orthogonal circles, must be a diameter of one of them.

20. If A, B, C, D are any four points, prove that the circles ABC, ABD cut at the same angle as the circles ACD, BCD.

21. If two circles intersect at O, and if POQ, P′OQ′ are two lines terminated by the circumferences, prove that PP′, QQ′ cut at the same angle as the circles.

22. PQ, RS are perpendicular diameters of two orthogonal circles; prove that PS is perpendicular to QR and cuts it at one of the common points of the two circles.

23. A variable circle cuts a fixed circle orthogonally and passes through a fixed point; prove that the locus of its centre is a straight line.

24. A variable circle is orthogonal to each of two fixed intersecting circles; prove that the locus of its centre is a straight line

CHAPTER IX.

POLES AND POLARS.

THE work of this chapter describes one method of establishing the form of correspondence alluded to in the discussion of the Principle of Duality in Chapter VII.

Definition.

If P, Q are points on the diameter AB of a circle, such that {AB ; PQ} is harmonic, and if QR is drawn parallel to the tangent at A, then QR is called the *polar* of P w.r.t. the circle, and P is called the *pole* of QR.

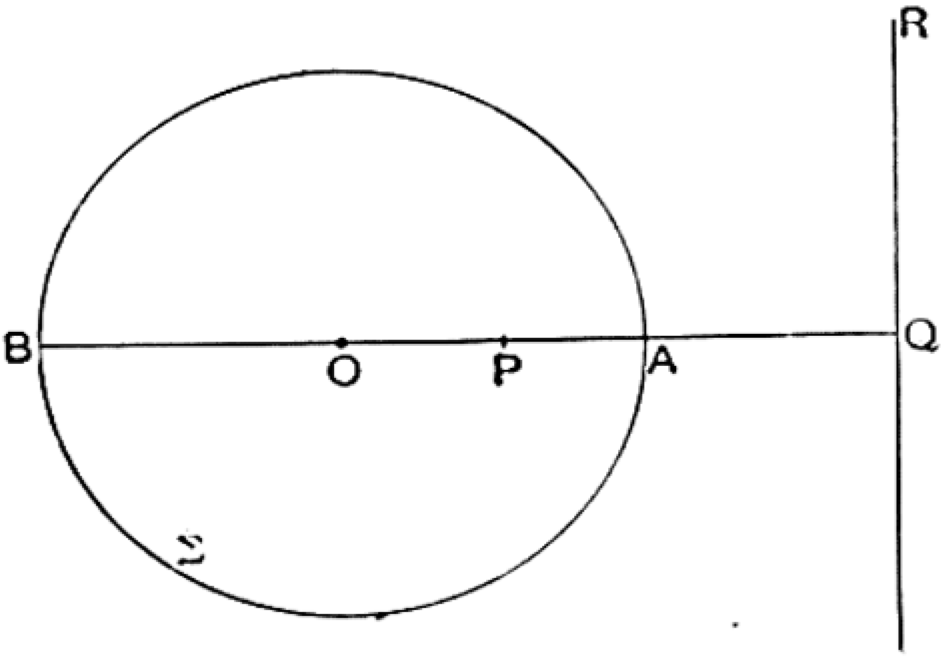

FIG. 57.

If O is the centre of the circle, it follows at once that

$$OP \cdot OQ = (\text{radius})^2,$$

and that OP is perpendicular to the polar of P.

Many of the polar properties of a circle admit of easy analytical treatment.

I. *The polar of* P(x', y') *w.r.t. the circle* $x^2+y^2=a^2$ *is* $xx'+yy'=a^2$.

The equation of OP is $\dfrac{x}{x'}=\dfrac{y}{y'}$.

∴ the polar of P, being perpendicular to OP, is of the form

$$xx'+yy'=k.$$

But the length of the perpendicular from O to this line is

$$\frac{k}{\sqrt{x'^2+y'^2}}=\text{OQ}=\frac{a^2}{\text{OP}}=\frac{a^2}{\sqrt{x'^2+y'^2}}; \quad \therefore\ k=a^2.$$ Q.E.D.

II. *If the polar of* P *passes through* R, *then the polar of* R *passes through* P.

Let the coordinates of P, R be (x_1, y_1), (x_2, y_2).

The polar of P is $xx_1+yy_1=a^2$: this passes through R.

$$\therefore\ x_2x_1+y_2y_1=a^2. \quad \text{.............................(1)}$$

But the polar of R is $xx_2+yy_2=a^2$.

∴ by (1) the point P, (x_1, y_1) lies on the polar of R. Q.E.D.

Definition.

If two points are so situated that the polar of either passes through the other w.r.t. a circle, then the points are called *conjugate points* w.r.t. the circle.

From II., (x_1, y_1) and (x_2, y_2) are conjugate points w.r.t. $x^2+y^2=a^2$, if $x_1x_2+y_1y_2=a^2$.

III. P *and* R *are conjugate points w.r.t. a circle*; *if* PH RK *are the tangents from* P, R *to the circle, then* $\text{PR}^2=\text{PH}^2+\text{RK}^2$.

With the same notation,

$$\begin{aligned}\text{PR}^2&=(x_1-x_2)^2+(y_1-y_2)^2\\&=x_1^2+y_1^2+x_2^2+y_2^2-2(x_1x_2+y_1y_2)\\&=\text{OP}^2+\text{OR}^2-2a^2=(\text{OP}^2-a^2)+(\text{OR}^2-a^2)\\&=\text{PH}^2+\text{RK}^2.\end{aligned}$$ Q.E.D.

IV. *Salmon's Theorem.* P, S are two points; PX, SY are the perpendiculars from P, S to the polars of S, P respectively w.r.t. a circle, centre O; then $\dfrac{\text{OP}}{\text{OS}}=\dfrac{\text{PX}}{\text{SY}}$.

This is a simple analytical exercise.

THEOREM 45.

(1) If the polar of P passes through R, then the polar of R passes through P.

(2) If a line p contains the pole of a line r, then r contains the pole of p.

(3) If P, R are the poles of p, r, then the pole of PR is the meet of p, r.

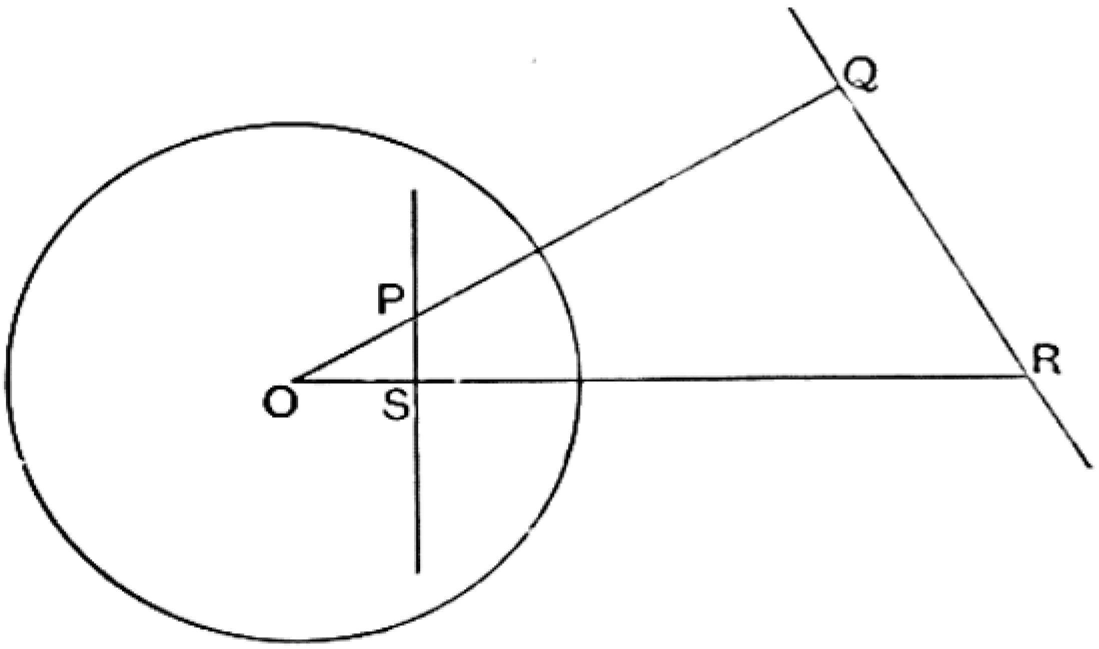

FIG. 58.

(1) Let O be the centre of the circle, and let OP meet the polar of P at Q; R is any point on the polar of P.

Draw PS perpendicular to OR.

Since $P\hat{S}R = 90° = P\hat{Q}R$, PQRS is concyclic.

$$\therefore \text{OR} . \text{OS} = \text{OQ} . \text{OP} = (\text{radius})^2 \quad \text{and} \quad O\hat{S}P = 90° ;$$

∴ PS is the polar of R. Q.E.D.

(2) Let P, R be the poles of p, r.

It is given that p (the polar of P) passes through R.

∴ by (1), r (the polar of R) passes through P (the pole of p).

Q.E.D.

(3) Let p, r meet at K.

The polar of P passes through K, therefore the polar of K passes through P. Similarly the polar of K passes through R.

∴ the polar of K is PR. Q.E.D.

Corollary.

(1) If any number of points are collinear, then their polars are concurrent.

(2) If any number of lines are concurrent, then their poles are collinear.

For, in (1), if P is a variable point on a fixed line l, whose pole is L, the polar of P passes through the fixed point L.

And in (2), if p is a variable line through a fixed point L, whose polar is l, then the pole of p lies on the fixed line l.

Definition.

If two lines are such that the pole of either w.r.t. a circle lies on the other, then the lines are called *conjugate lines* w.r.t. the circle.

For "*conjugate points*," see page 94.

1. If the point P is outside the circle, prove that the chord of contact of the tangents from P is the polar of P.

2. What is the polar of a point on the circle ?
What is the polar of the centre of the circle ?
What is the pole of a diameter of the circle ?

3. PR is the diameter of a circle Σ, orthogonal to a circle S; O is the centre of S; OP meets Σ at Q; prove that QR is the polar of P w.r.t. S.

4. If P, R are conjugate points w.r.t. a circle Σ, prove that the circle on PR as diameter is orthogonal to Σ.

5. A variable chord PQ of a fixed circle passes through a fixed point; prove that the tangents at P, Q intersect on a fixed line.

6. From a variable point T on a fixed line, tangents TP, TQ are drawn to a fixed circle; prove that PQ passes through a fixed point.

7. The tangent at A to the circumcircle of the triangle ABC meets BC at T and is produced to U so that AT = TU; prove that A, U are conjugate points w.r.t. any circle through B, C.

8. P, R are conjugate points w.r.t. a circle, centre O; prove that the pole of PR is the orthocentre of the triangle OPR.

9. The centre A of a circle S lies on a circle Σ; CD is their common chord; any line through A cuts CD at H and Σ at K; prove that H, K are conjugate points w.r.t. S.

10. Prove that the angle between two lines is equal to the angle which their poles subtend at the centre of the circle.

11. If two circles are orthogonal, prove that the extremities of any diameter of one are conjugate points w.r.t. the other.

12. I is the incentre of the triangle ABC; perpendiculars through I to IA, IB, IC meet any tangent to the incircle at P, Q, R; by determining the poles of AP, BQ, CR, prove that AP, BQ, CR are concurrent.

13. **Prove that the locus of a point whose polars w.r.t. three given circles are concurrent is the circle orthogonal to each of the given circles.**

14. ABC is a triangle, right-angled at B; BB′ is a median. Prove that A, C are conjugate points w.r.t. any circle which touches BB′ at B.

15. PQ is the chord of contact of the tangents from T to a circle: the tangent at any other point R on the circle meets PQ at S; prove that TR is the polar of S.

16. P, Q are two points subtending 90° at a point C on a circle: if the tangent at C bisects PQ, prove that P, Q are conjugate points.

17. Assuming the theorem that if the incircle of the triangle ABC touches BC, CA, AB at X, Y, Z, then AX, BY, CZ are concurrent, deduce by Theorem 45, corollary, a theorem for the circumcircle of a triangle.

18. **PQR is a triangle: P′, Q′, R′ are the poles of QR, RP, PQ w.r.t. a circle; prove that P, Q, R are the poles of Q′R′, R′P′, P′Q′.**

Definitions.

(1) If two triangles are such that the vertices of either are the poles of the sides of the other w.r.t. a circle, then the triangles are called *conjugate triangles* w.r.t. the circle.

(2) If a triangle is conjugate to itself (*i.e.* each vertex is the pole of the opposite side), it is called a *self-conjugate* or *self-polar* triangle.

19. P, Q are any two conjugate points and R is the pole of PQ; prove that PQR is a self-conjugate triangle.

20. If a triangle is self-conjugate w.r.t. a circle, prove that its orthocentre is the centre of the circle.

21. **Given a triangle, prove that there exists one, and only one, circle (real or imaginary) w.r.t. which it is self-conjugate: and that the circle is real if, and only if, the triangle is obtuse angled.**

Definition.

The circle for which a triangle is self-conjugate is called the *polar circle* of the triangle.

It is left to the reader to prove that, with the usual notation, the polar circle of the triangle ABC is the circle, centre H, and radius $\sqrt{\text{HA}\,.\,\text{HD}}$.

THEOREM 46.

If P is the pole of a line QR w.r.t. a circle Σ, then any line through P is cut harmonically by P, QR and Σ.

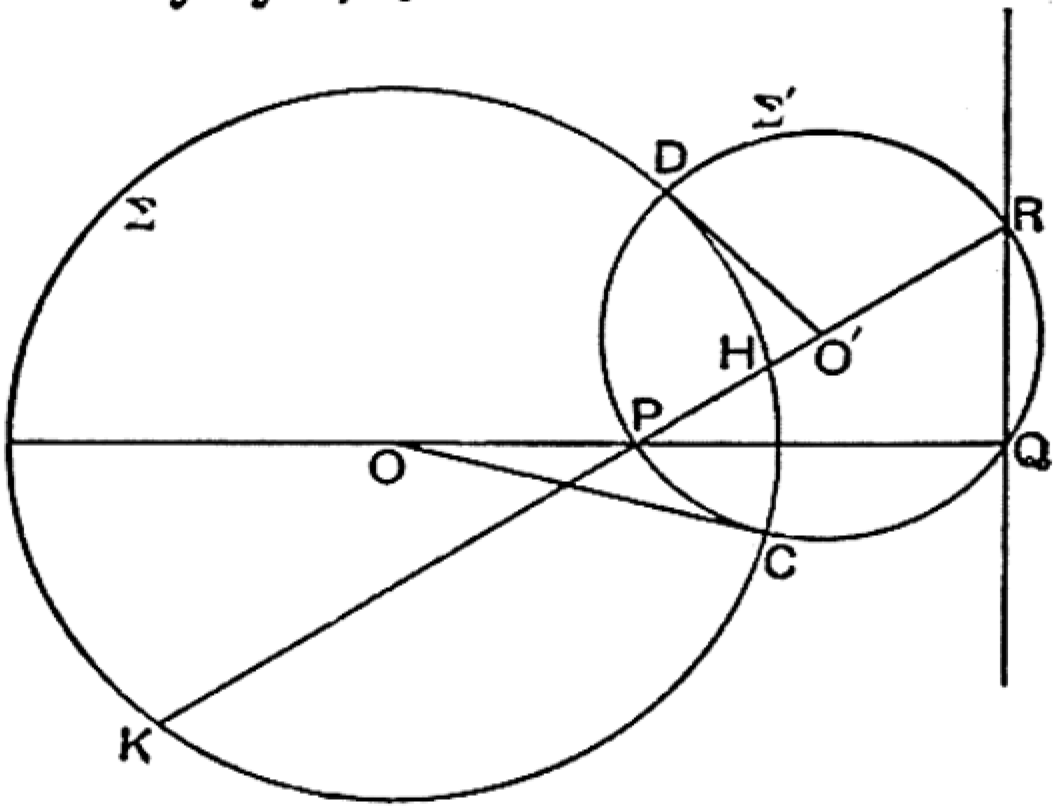

FIG. 59.

Let O be the centre of Σ: and let OP meet the polar of P at Q. PR is any line through P cutting Σ at H, K.

On PR as diameter, describe a circle Σ', and let it cut Σ at C, D. Let O′ be the centre of Σ'.

Since $P\hat{Q}R = 90°$, Σ' passes through Q.

Since P is the pole of QR, $OP \cdot OQ = OC^2$.

$\therefore$ Σ is orthogonal to Σ'.

$\therefore$ O′D is a tangent to Σ.

$\therefore$ $O'H \cdot O'K = O'D^2 = O'P^2 = O'R^2$.

$\therefore$ {RP ; HK} is harmonic. Q.E.D.

Corollary.

If P, R are conjugate points w.r.t. a circle, and if PR meets the circle at H, K, then {PR ; HK} is harmonic.

The corresponding dual property is proved in the next theorem.

THEOREM 47.

If *p*, *r* are two conjugate lines meeting at a point T outside the circle, and if TX, TY are the tangents to the circle, then *p*, *r* are harmonically conjugate TX, TY.

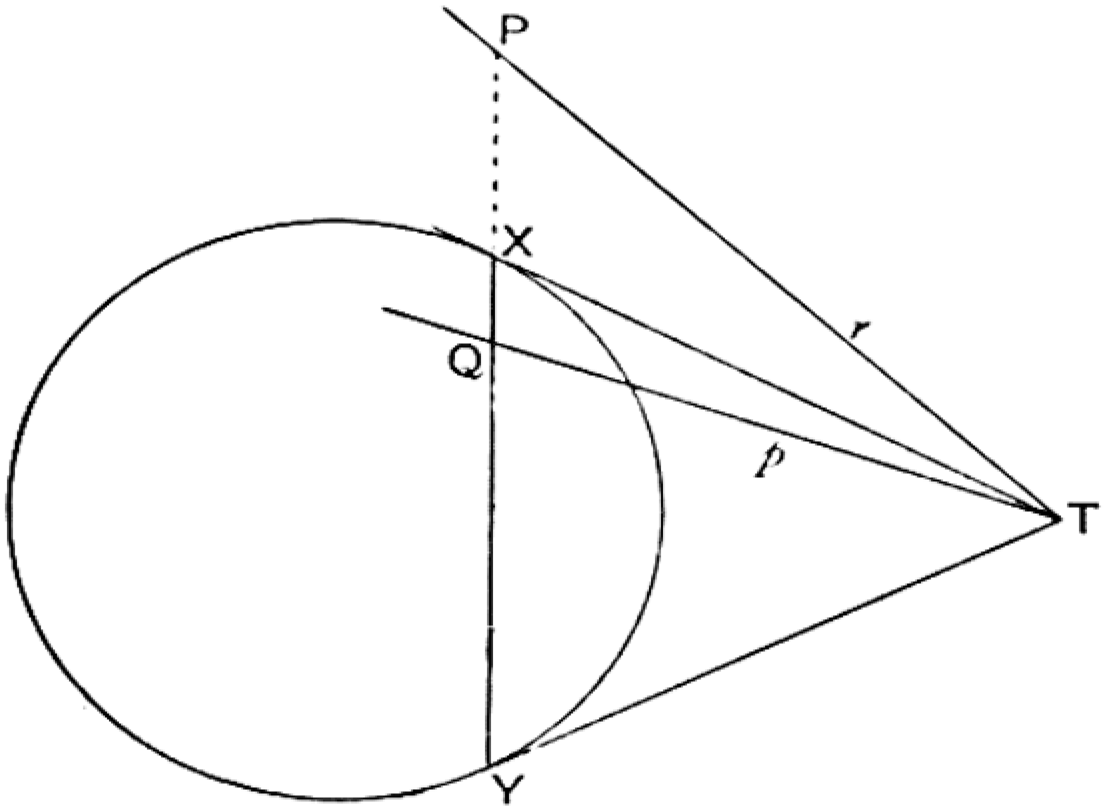

FIG. 60.

Let P be the pole of p; then it is given that P lies on r.

Now the polar of P (*i.e.* p) passes through T; therefore the polar of T (*i.e.* XY) passes through P.

Let PXY cut p at Q.

Then {PQ ; XY} is harmonic. [Th. 46.]

T{PQ ; XY} is harmonic. Q.E.D.

THEOREM 48.

If four points form a harmonic range, then their polars form a harmonic pencil.

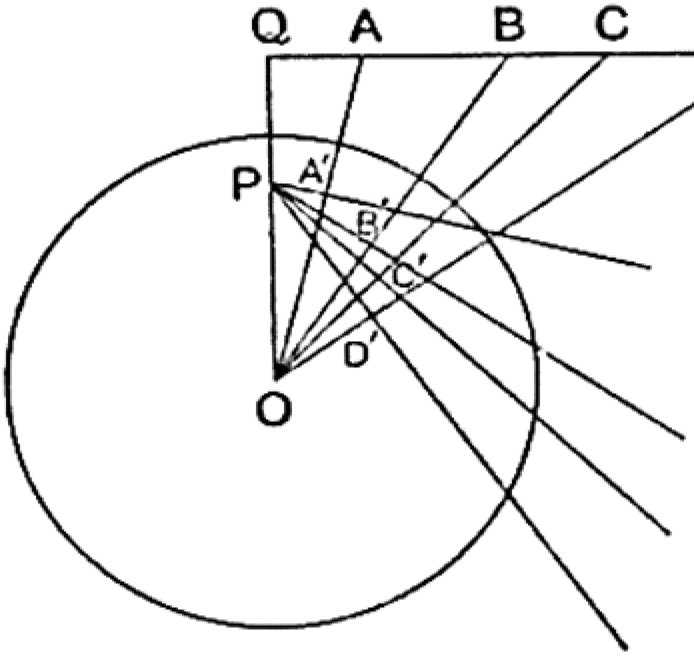

FIG. 61.

Let {ABCD} be harmonic, and let P be the pole of AD.

Let O be the centre of the circle; OP cuts AD at Q.

Draw PA′, PB′, PC′, PD′ perpendicular to OA, OB, OC, OD.

The polar of P passes through A; therefore the polar of A passes through P. But PA′ is perpendicular to OA; therefore PA′ is the polar of A.

Similarly PB′, PC′, PD′ are the polars of B, C, D.

Now the rays of the pencil P{A′B′C′D′} are perpendicular to the rays of the pencil O{ABCD}; therefore these pencils are equiangular.

But O{ABCD} is harmonic;

∴ P{A′B′C′D′} is harmonic. Q.E.D.

Corollary.

The cross-ratio of any four collinear points is equal to the cross-ratio of the pencil formed by their polars.

22. A variable chord PQ of a fixed circle passes through a fixed point A and is produced to R so that $RP . RQ = RA^2$; prove that the locus of R is a straight line.

23. The tangents at the extremities of a chord PQ of a circle meet at T; prove that any other tangent to the circle is cut harmonically by TP, TQ; PQ and its point of contact.

24. P is the pole of the chord AB and O the mid-point of AB: any line through P cuts the circle at C, D; prove that AB bisects $C\hat{O}D$.

25. Through the mid-point M of a chord AB, two straight lines MP, MQ are drawn to the circumference so that AB bisects internally $P\hat{M}Q$; prove that AB and the tangents at P, Q are concurrent.

26. A chord HK passes through the pole T of a chord PQ; PV is a chord parallel to HK; prove that QV bisects HK. [Prove V{PQ; HK} is harmonic.]

27. C is the mid-point of a chord AB; PQ is another chord through C; H is the pole of AB; prove that $P\hat{H}C = Q\hat{H}C$.

28. T is a variable point on a fixed line; TP, TQ are tangents to a fixed circle; PN, QM are the perpendiculars to the fixed line; prove that $\frac{1}{PN} + \frac{1}{QM}$ is constant.

29. If two chords AB, CD of a circle are conjugate lines, prove that ACBD form a harmonic system of points on the circle.

30. PQ, PR are two chords of a circle; the perpendicular bisector of PQ meets PR at H; prove that the line joining H to the pole of QR is parallel to PQ.

31. The tangent at a point X on a circle meets a chord AB at P; O is the mid-point of AB; if the polar of P cuts AB at R, prove that PX touches the circle ORX.

32. If a given quadrilateral ABCD is circumscribed to a given circle, and if AP, BQ, CR, DS are the perpendiculars to a variable tangent, prove that $\frac{AP \cdot CR}{BQ \cdot DS}$ is constant. [Use Salmon's theorem.]

33. The incircle of the triangle ABC touches BC, CA, AB at X, Y, Z; prove that the diameter through X of the incircle meets YZ on the line joining A to the mid-point A′ of BC. [Prove that AA′ is conjugate to the line through A parallel to BC.]

34. Two chords AB, CD are conjugate lines w.r.t. the circle; P is any point on the circle; CD cuts PA, PB at H, K; prove that the polar of H passes through K.

35. Two chords AB, CD are conjugate lines w.r.t. the circle: any line through A cuts the circle at P and CD at Q; prove that B{CD; PQ} is harmonic.

36. PQ is a chord of a circle bisecting the chord AB at C; the tangents at P, Q meet AB at H, K; prove that CH = CK.

37. Two chords AB, CD of a circle are conjugate lines w.r.t. the circle; prove that AC . BD = BC . AD = ½AB . CD.

38. A circle through A, B intersects CA, CB at E, F; V is the mid-point of EF; prove that CV, AB are conjugate lines w.r.t. the circle ABC.

39. The incircle of the triangle ABC touches BC, CA, AB at X, Y, Z; PX is a diameter of the incircle; PA, PY, PZ cut BC at M, Q, R; prove that QM = MR.

40. T is the pole of BC w.r.t. the circumcircle of ABC; a line through T parallel to the tangent at A cuts AB, AC at D, E: prove that DT = TE.

THE INSCRIBED QUADRANGLE

THEOREM 49.

(1) If a quadrangle ABCD is inscribed in a circle, its diagonal point triangle EFG is a self-conjugate triangle.

(2) The tangents at A, B, C, D form a quadrilateral having for its diagonal line triangle the diagonal point triangle of the quadrangle ABCD.

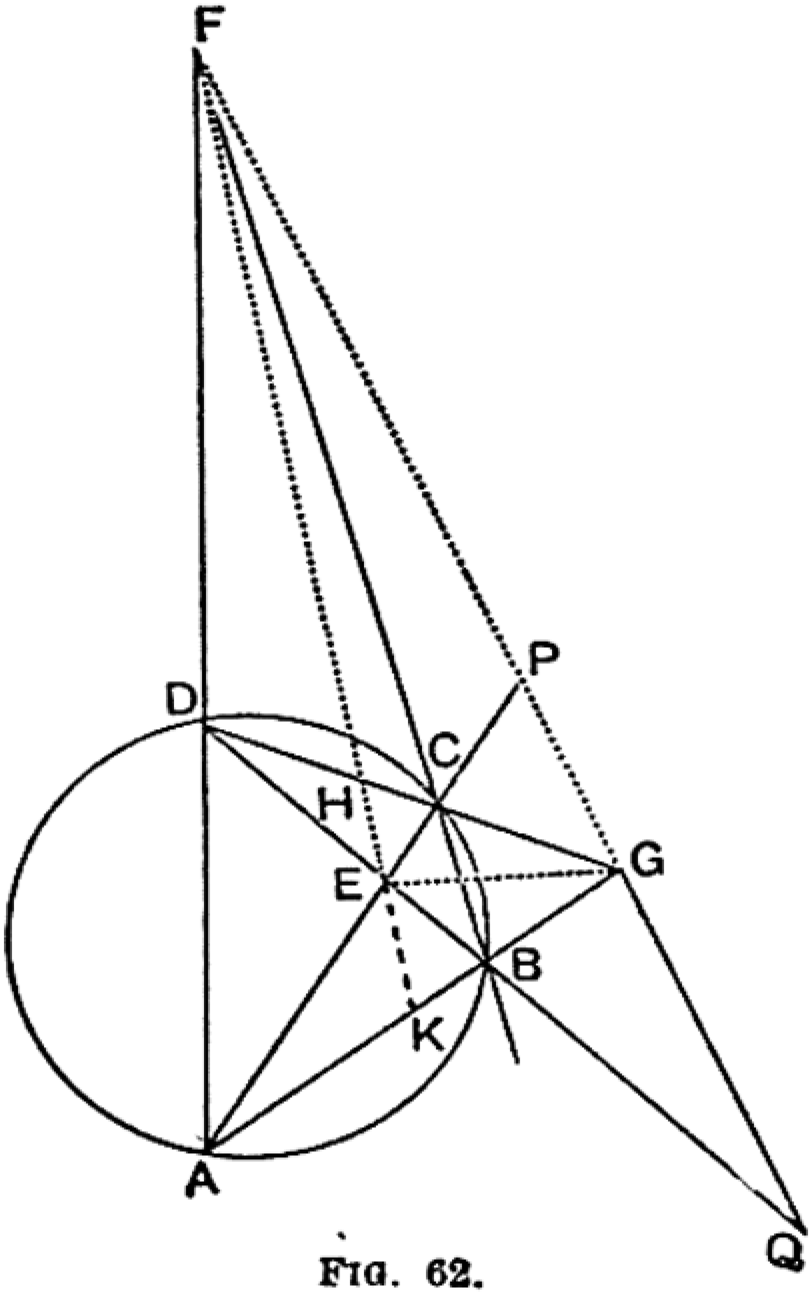

FIG. 62.

(1) {AECP} is harmonic; ∴ the polar of E passes through P.

{DEBQ} is harmonic; ∴ the polar of E passes through Q.

∴ the polar of E is PQ or FG.

Since {DHCG} and {AKBG} are harmonic, the polar of G is HK or EF, and similarly the polar of F is GE.

∴ EFG is a self-conjugate triangle. Q.E.D.

(2) [The reader should draw his own figure and draw in the tangents at A, B, C, D.]

BC contains the pole (F) of EG; ∴ EG contains the pole of BC.

∴ the tangents at B, C meet on EG.

Similarly the tangents at A, D meet on EG; those at C, D meet on EF: those at A, B meet on EF; those at B, D meet on FG; those at A, C meet on FG.

∴ EF, FG, GE are diagonal lines of the quadrilateral formed by the tangents at A, B, C, D. Q.E.D.

THE CIRCUMSCRIBED QUADRILATERAL.

THEOREM 50.

(1) If a quadrilateral $abcd$ is circumscribed about a circle, its diagonal line triangle efg is a self-conjugate triangle.

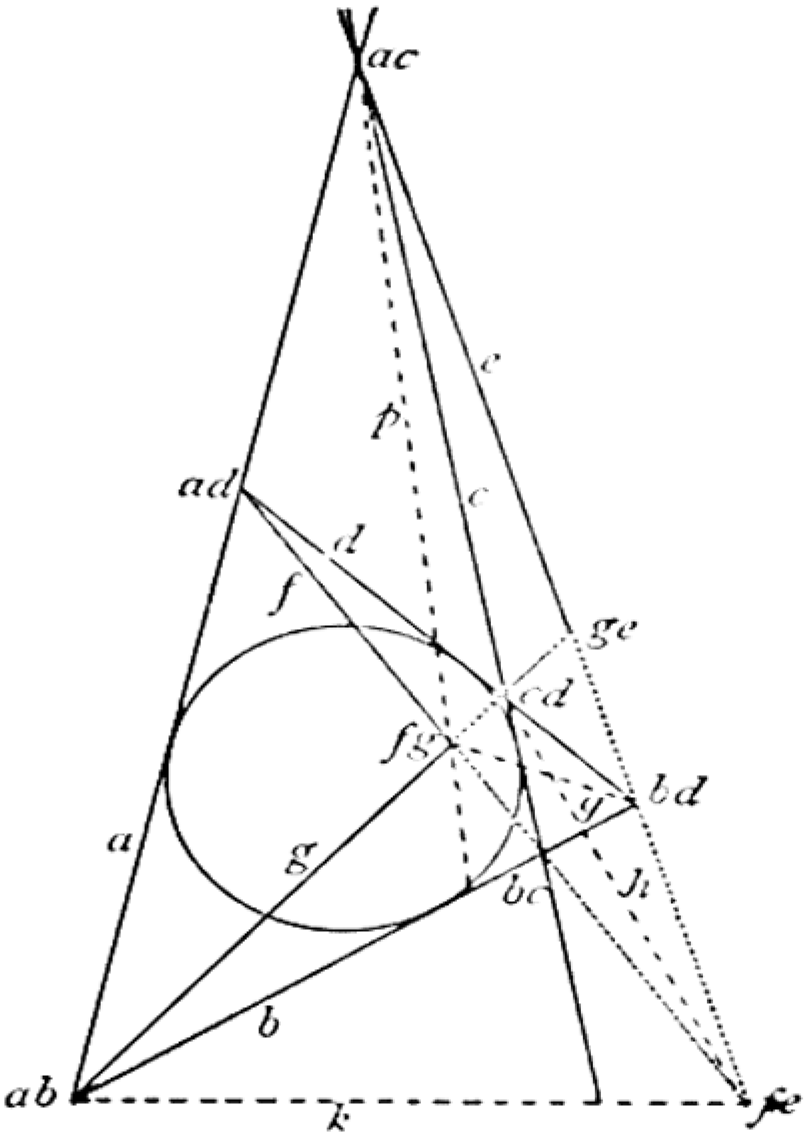

FIG. 63.

(2) The points of contact of a, b, c, d form a quadrangle having for its diagonal point triangle the diagonal line triangle of the quadrilateral a, b, c, d.

(1) $\{aecp\}$ is harmonic; ∴ the pole of e lies on p.

$\{debq\}$ is harmonic; ∴ the pole of e lies on q.

∴ the pole of e is pq or fg.

Since $\{dhcg\}$ and $\{akbg\}$ are harmonic, the pole of g is hk or ef, and similarly the pole of f is ge.

∴ efg is a self-conjugate triangle. Q.E.D.

(2) [The reader should draw his own figure and join up the points of contact of a, b, c, d.]

bc lies on the polar (f) of eg; ∴ eg lies on the polar of bc.

∴ the join of the points of contact of b, c passes through eg.

Similarly the join of the points of contact of a, d passes through eg; that of c, d passes through ef; that of a, b passes through ef; that of b, d passes through fg; that of a, c passes through fg.

∴ ef, fg, ge are diagonal points of the quadrangle formed by the points of contact of a, b, c, d. Q.E.D.

41. PQ is a variable diameter of a fixed circle; A is any fixed point. Find the locus of the orthocentre of triangle APQ.

42. P is a variable point on the circumcircle of the fixed triangle ABC; AP, BP, CP meet BC, CA, AB at F, G, H; prove that each side of the triangle FGH passes through a fixed point.

43. PQ is a variable diameter of a fixed circle; A is any fixed point; AP, AQ cut the circle at R, S; prove that RS passes through a fixed point.

44. Using a ruler only, construct the polar of a given point w.r.t. a given circle.

45. Using a ruler only, construct the pole of a given line w.r.t. a given circle.

46. Using a ruler only, construct the tangent to a given circle at a given point.

47. If, in Fig. 62, PEQ is a chord through E parallel to FG, and if O is the centre of the circle, prove that $O\hat{E}P = 90^\circ$.

48. A, B are two fixed points; PQ is a variable chord of a fixed circle and passes through A; BP, BQ meet the circle at R, S; prove that RS passes through a fixed point. Enunciate the dual theorem.

49. In Fig. 62, prove that the circle on AC as diameter is orthogonal to the circle PEG.

50. AB, CD are chords of a circle intersecting at O; the tangents at A, D meet at P; the tangents at B, C meet at Q; prove that P, O, Q are collinear.

51. Two sides of a triangle self-conjugate to a given circle cut the circle at four points forming a quadrangle ABCD; prove that the bisectors of the angles ABC, ADC meet on AC.

52. Through a fixed point O are drawn two variable chords POQ, ROS of a fixed circle; find the locus of the meet of PR, QS.

53. If a circle circumscribes a quadrangle, prove that its centre is the orthocentre of the diagonal point triangle. Enunciate the dual theorem.

54. A quadrangle is inscribed in a circle Σ; prove that the circles whose diameters are the sides of the diagonal point triangle are orthogonal to Σ.

55. What is the common self-conjugate triangle of two non-intersecting circles?

56. AB, BC, CD, DA are four tangents to a circle, centre O, and A′, B′, C′, D′ are their points of contact; if A′D′, B′C′ meet at P, prove that PO is perpendicular to AC.

CHAPTER X.

INVERSION.

Definition.

If on the line joining a given point O to a point P, a point P′ is taken such that OP . OP′ $= k^2$, where k is a given constant, then P′ is called the *inverse point* of P w.r.t. the centre or origin of inversion O or w.r.t. the circle, centre O, radius k, which is called the *circle of inversion*; k is called the *radius of inversion.*

If the point P traces out a curve Σ, then the point P′ will trace out a curve Σ', which is called the *inverse curve* of Σ w.r.t. O, radius k.

This definition of inverse points agrees with that given on page 88.

If O is fixed, but k receives different values, the *position* and *size* of Σ' is altered, but the *shape* remains the same; in other words, the various curves Σ' will all be homothetic to each other. Consequently, in general, the actual numerical value of k is immaterial and is not specified. In such cases, we shall speak of inverting w.r.t. O, and leave out all mention of k.

In order to obtain a general idea of the process, the reader should take (i) groups of isolated points, (ii) various curves, and draw freehand the rough shapes of their inverses w.r.t. any selected origin

THEOREM 51.

(1) If a straight line is inverted w.r.t. a point O on it, the inverse is the same straight line.

(2) If a straight line is inverted w.r.t. a point O outside it, the inverse is a circle through O.

(3) If a circle is inverted w.r.t. a point O on it, the inverse is a straight line.

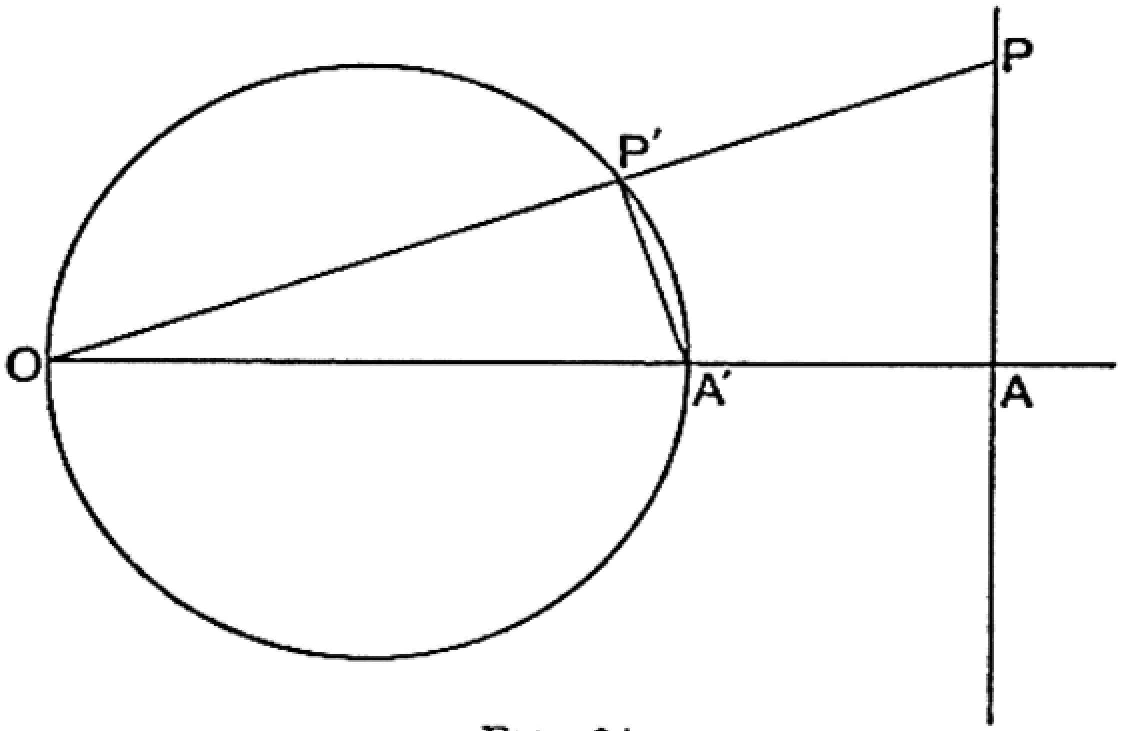

FIG. 64.

(1) By definition, each point on the line inverts into another point on the same line. Q.E.D.

(2) From O, draw OA perpendicular to the line; take any point P on the given line, and let A′, P′ be the inverses of A, P w.r.t. O.

Since OA . OA′ = OP . OP′, A′APP′ is a cyclic quadrilateral.

$$\therefore \; O\hat{P}'A' = O\hat{A}P = 90^\circ.$$

∴ P′ lies on the circle on OA′ as diameter.

∴ the inverse of the straight line is a circle through O whose centre lies on the perpendicular from O to the line.

(3) Let OA′ be the diameter through O of the given circle. Take any point P′ on the circumference, and let A, P be the inverses of A′P′ w.r.t. O.

Since A′, A, P, P′ are concyclic, $P\hat{A}A' = O\hat{P}'A' = 90^\circ$, since OA′ is a diameter.

∴ P lies on a line through A perpendicular to OA.

∴ the inverse of the locus of P′ is a straight line. Q.E.D.

THEOREM 52.

If a circle is inverted w.r.t. a point O which does not lie on the circle, then the inverse is another circle.

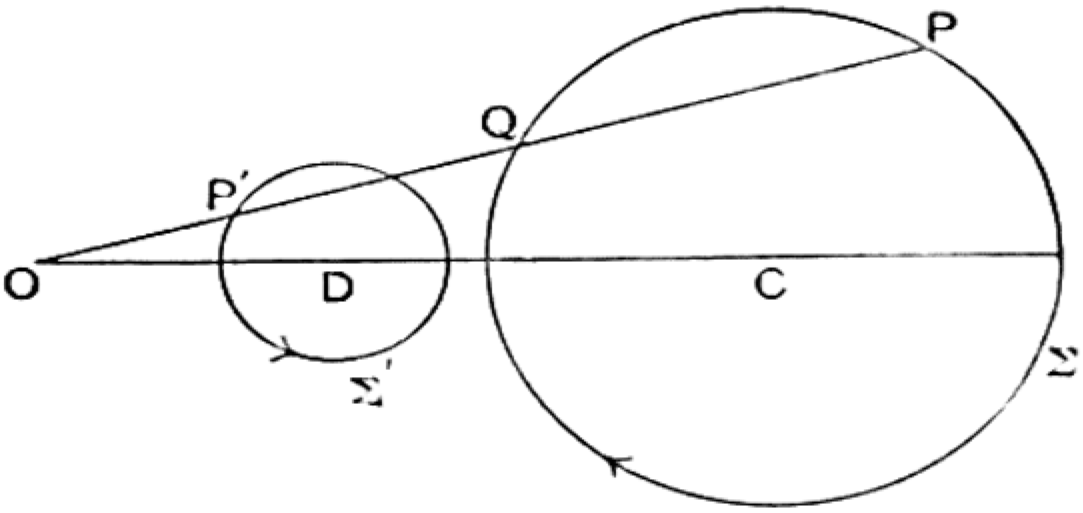

FIG. 65.

Let P be any point on the given circle Σ, and let OP meet Σ again at Q. Let P′ be the inverse of P.

Then OP . OP′ $=k^2$, where $k=$ radius of inversion, and OP . OQ $=t^2$, where $t=$ tangent from O to Σ.

$$\therefore \frac{OP'}{OQ}=\frac{k^2}{t^2}=\text{constant}.$$

∴ as Q traces out Σ, P′ traces out a homothetic circle Σ′. Q.E.D.

It is important to notice that the centre D of Σ′ is *not* the inverse of the centre C of Σ. [See Ex. 6.]

Notice also that P′ describes Σ′ in the opposite direction to that in which P describes Σ, if O lies outside Σ.

The reader should also consider what difference there is in the proof, if O lies inside Σ.

It is clear from the proof that O is a centre of similitude of Σ and Σ′.

THEOREM 53.

The angle at which two curves cut is equal to the angle at which their inverse curves cut at the corresponding point of intersection.

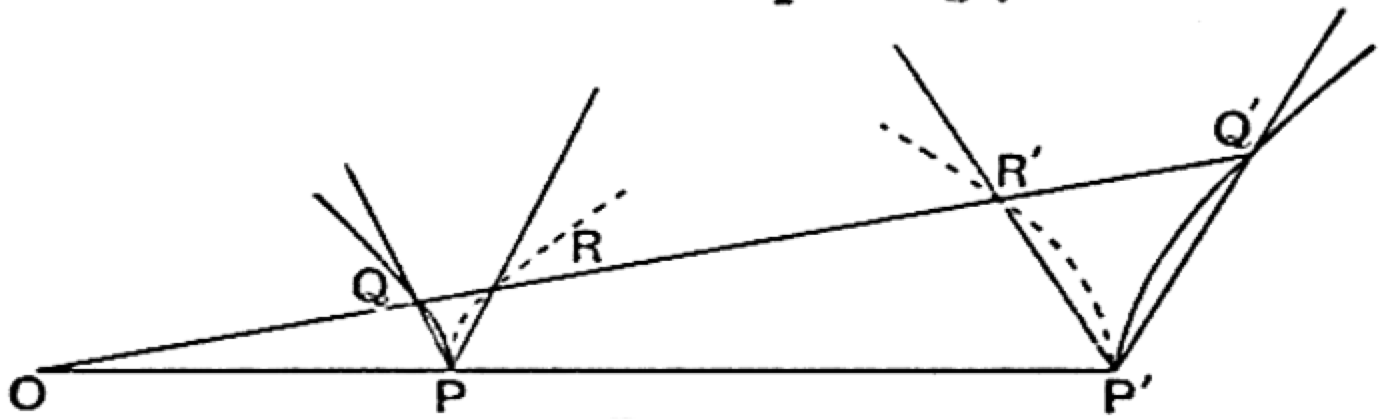

FIG. 66.

O is the centre of inversion; P, P′ are the corresponding points of intersection of the two curves. Through O, draw any line OQRR′Q′ cutting the two given curves at Q, R and their inverses at Q′, R′.

Since $OQ \cdot OQ' = OP \cdot OP' = OR \cdot OR'$, PQQ′P′ and PRR′P′ are cyclic quadrilaterals.

$$\therefore\ O\hat{P}Q = O\hat{Q}'P' \quad \text{and} \quad O\hat{P}R = O\hat{R}'P'.$$

$$\therefore\ Q\hat{P}R = Q'P'R'.$$

Now, when the line OQ moves up to the line OP, the limiting positions of PQ, PR and P′Q′, P′R′ are those of the tangents at P and P′ to the curves.

∴ the angle of intersection is unaltered by inversion. Q.E.D.

In particular, note the following:

(i) Orthogonal circles in general invert into orthogonal circles.

(ii) If two orthogonal circles intersect at A, their inverses w.r.t. A are two perpendicular straight lines.

(iii) If two orthogonal circles are inverted w.r.t. a point on one of them, the inverse is a circle and a diameter of that circle.

(iv) If two curves touch each other, the inverse curves touch each other.

The process of inversion constitutes one form of correspondence between two geometrical systems: and to any property which the first possesses, there must correspond a property of the second. The process therefore will sometimes yield a proof of a property by showing that it corresponds to some known (simpler) property. An example is given to illustrate this method.

Example.

Three circles S_1, S_2, S_3 cut one another orthogonally: if their common chords are AB, CD, EF, the circles ACE, ADF touch at A.

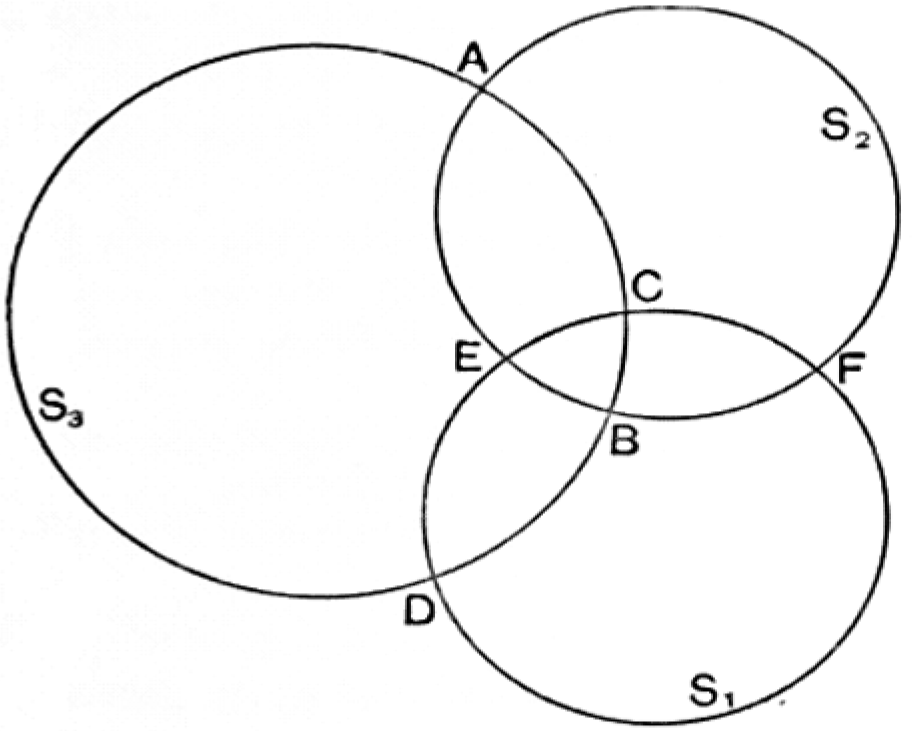

FIG. 67.

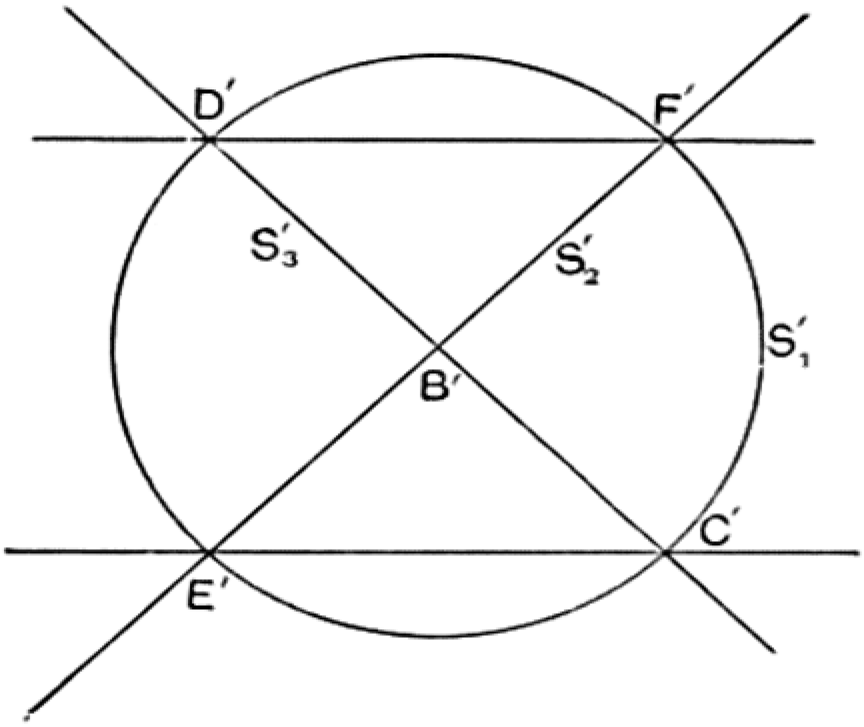

FIG. 68.

Invert w.r.t. A and denote by dashes corresponding points, lines or circles in the inverse figure. [To avoid confusion, the inverse figure is drawn by itself: actually, of course, the two figures are really superimposed.]. S_2 and S_3 become straight lines, S_2', S_3' orthogonal to the circle S_1', and are therefore diameters of S_1', so that B′ is the centre of S_1'.

Now the circles ACE, ADF become the straight lines C′E′, D′F′, which are from symmetry parallel to each other.

∴ the circles ACE, ADF touch each other at A. Q.E.D.

1. What is the inverse of a circle w.r.t. its centre? What is the inverse of two parallel straight lines?

2. Regarding a straight line as the limiting form of a circle, when the centre is at infinity, find the inverse of a point w.r.t. a straight line.

3. What is the inverse of a parallelogram w.r.t. any point?

4. A system of circles all pass through two points A, B; what is the inverse of the system w.r.t. A?

5. Invert w.r.t. O the following known theorems:

(i) From any point outside a circle, two lines can be drawn to touch a given circle OAB.

(ii) OBC is a triangle; a parallel to BC cuts OB, OC at D, E; then $\frac{OD}{OB}=\frac{OE}{OC}$

(iii) O is a fixed point and P a variable point on a fixed line; OP is produced to Q, so that $\frac{\text{OP}}{\text{OQ}}$ is constant; then the locus of Q is a parallel line.

6. With the notation of Theorem 52, prove that $\frac{\text{OC}}{\text{OD}}=\frac{t^2}{k^2}$, and hence show that C, D cannot be inverse points.

7. O is any point on the common chord of two circles; prove that, by inverting w.r.t. O, each circle can be inverted into itself.

8. What is the inverse of two circles and their four common tangents?

9. P, Q, R are three points on a circle, centre A; PQ, PR are produced to H, K so that PQ . PH = PR . PK; prove that PA is perpendicular to HK.

10. A, B, C are three collinear points. Two variable equal circles PAB, PBC meet at P; prove that the locus of P is a straight line.

11. Prove that the inverse of a plane w.r.t. a point O outside it is a sphere through O.

12. Prove that the inverse of a sphere is either a plane or a sphere.

13. Prove that the inverse of a circle w.r.t. a point O not in its plane is another circle.

14. If C is the mid-point of AB, and if A′, B′, C′ are the inverses of A, B, C w.r.t. any point O on AB, prove that {OC′; A′B′} is harmonic.

15. PQ, RS are common tangents to two circles PAR, QAS; prove that the circles PAQ, RAS touch each other.

16. A′, B′, C′, D′ are the inverses of A, B, C, D; if {ABCD} is harmonic, prove that A′B′C′D′ form either a harmonic range or a harmonic system of points on a circle. [Use Theorem 44.]

17. T is a variable point on a circle Σ_1: PQ is the chord of contact of the tangents from T to another fixed circle Σ_2; prove that the locus of the mid-point of PQ is in general a circle. What is the exceptional case?

18. B is a point on the base AC of the triangle OAC; prove that the circumcentres of the triangles OBC, OCA, OAB are concyclic with O.

19. A is a fixed point, PAQ is an angle of fixed size and the triangle APQ is of constant area. If P lies on a fixed circle, find the locus of Q.

20. Three circles BOC, COA, AOB are such that the centres of BOC, COA lie on OA, OB; prove that the centre of AOB lies on OC.

21. A circle S passes through the centre of a circle Σ: the external common tangents to S and Σ touch S at P, Q and meet at O. Prove that PQ touches Σ.

22. O, A, B, C, H are five points such that the circles OAB, OCH, and also the circles OBC, OAH are orthogonal; prove that the circles OCA, OBH are also orthogonal.

23. Prove that if a straight line is cut harmonically by two orthogonal circles, it must be a diameter of one. (Use Ex. 16.)

24. What is the inverse of two circles and the line joining their centres?

25. Two lines are drawn from a fixed point and contain a constant angle and cut a fixed line; prove that the circumcircle of the triangle so formed touches a fixed circle.

26. A variable circle touches a fixed circle and cuts another fixed circle orthogonally; prove that it also touches another fixed circle.

27. A, B are fixed points on a fixed circle Σ, P is a variable point on Σ; O is any other fixed point. Prove that the circles OAP, OBP cut at a constant angle.

28. Invert w.r.t. O the theorems:

(i) O is a fixed point on the fixed circle on AB as diameter, P is a variable point on the circle; then $A\hat{P}B = 90°$.

(ii) OP, OQ are variable chords drawn through a fixed point O on a fixed circle, and are equally inclined to a fixed chord OA; then PQ is fixed in direction.

(iii) It is possible to draw an unlimited number of concurrent straight lines cutting a given circle orthogonally.

(iv) The opposite angles of the cyclic quadrilateral OABC are supplementary.

29. If the circumcircles of the triangles ABC, ABD are orthogonal, prove that the circles CAD, CBD are also orthogonal.

30. A, B are two fixed points on a circle; PQ is a variable diameter; AP meets BQ at R; prove that the locus of R is a circle through A, B orthogonal to the given circle.

31. Two circles intersect orthogonally at P; O is any point on any circle touching the former circles at Q, R; prove that the circles OPQ, OPR intersect at an angle of 45°.

32. Explain the fallacy in the following argument: AB and CD are any two straight lines: O is a point on neither: AB and CD invert w.r.t. O into two circles intersecting at O. But O is the inverse of a point at infinity; therefore the original lines AB, CD intersect at infinity and are therefore parallel.

33. A variable sphere is described through a fixed point to touch two fixed spheres; prove that the locus of each point of contact is a circle.

34. Four intersecting lines form four triangles; prove that their circumcircles are concurrent, at P say, and that their circumcentres and P lie on a circle.

35. A′, B′, C′ are the inverses of three fixed points A, B, C w.r.t. a variable circle, centre P. Find the locus of P if $A'B'C'$ is a right-angled triangle.

36. Two circles in different planes both touch the line of intersection of the planes at the same point. Show that if a variable plane touches both circles (at P, Q say), it passes through a fixed point O, and OP . OQ is constant.

37. ABD, CAE, BCF are three circles touching each other at A, B, C; the common tangent at C passes through D, and DAE, DBF are straight lines; prove that EF touches the circles at E and F.

THEOREM 54.

P′ and Q′ are the inverse points of P, Q w.r.t. O ; p_1, p_2 are the lengths of the perpendiculars from O to PQ and P′Q′; then

$$\frac{P'Q'}{PQ}=\frac{p_2}{p_1}=\frac{k^2}{OP\,.\,OQ}=\frac{OP'\,.\,OQ'}{k^2}$$

where k is the radius of inversion

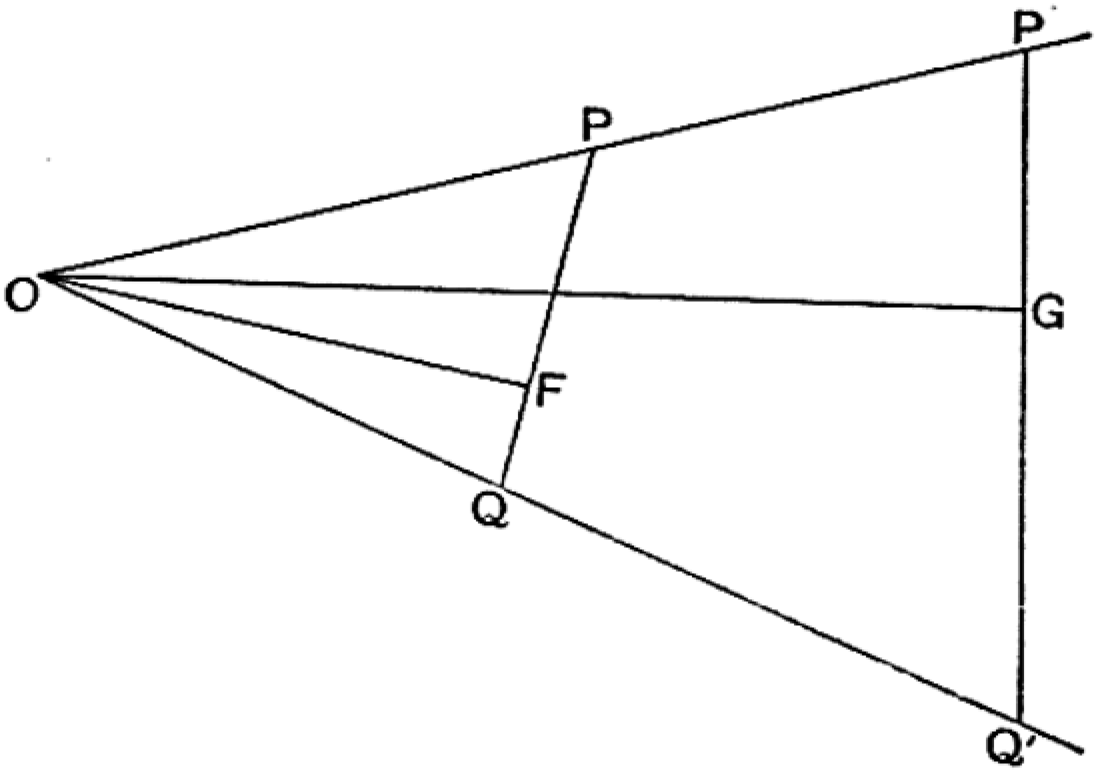

FIG. 69.

Draw OF, OG perpendicular to PQ, P′Q .

Since P, Q, Q′, P′ are concyclic, $O\hat{P}Q = O\hat{Q}'P'$ and $O\hat{Q}P = O\hat{P}'Q'$;
∴ the triangles OPF, OQ′G, and also OPQ, OQ′P′ are similar.

$$\therefore\ \frac{OG}{OF}=\frac{OQ'}{OP}=\frac{P'Q'}{PQ}\quad\text{or}\quad\frac{P'Q'}{PQ}=\frac{p_2}{p_1}.$$

Further,

$$\frac{P'Q'}{PQ}=\frac{OQ'}{OP}=\frac{OQ\,.\,OQ'}{OQ\,.\,OP}=\frac{OQ'\,.\,OP'}{OP\,.\,OP'}$$

$$=\frac{k^2}{OQ\,.\,OP}=\frac{OP'\,.\,OQ'}{k^2}.$$

Q.E.D.

By means of this theorem, metrical properties in one figure can be transformed into metrical properties in the inverse figure. Note in particular Ex. 40.

THEOREM 55.

(1) If a circle and two inverse points are inverted w.r.t. any point, then the inverse system is a circle and two inverse points.

(2) If C_1 and C_2 are inverse curves w.r.t. a circle Σ, and if the system is inverted w.r.t. any point so that C_1, C_2, Σ become C_1', C_2', Σ', then C_1' and C_2' are inverse curves w.r.t. the circle Σ'.

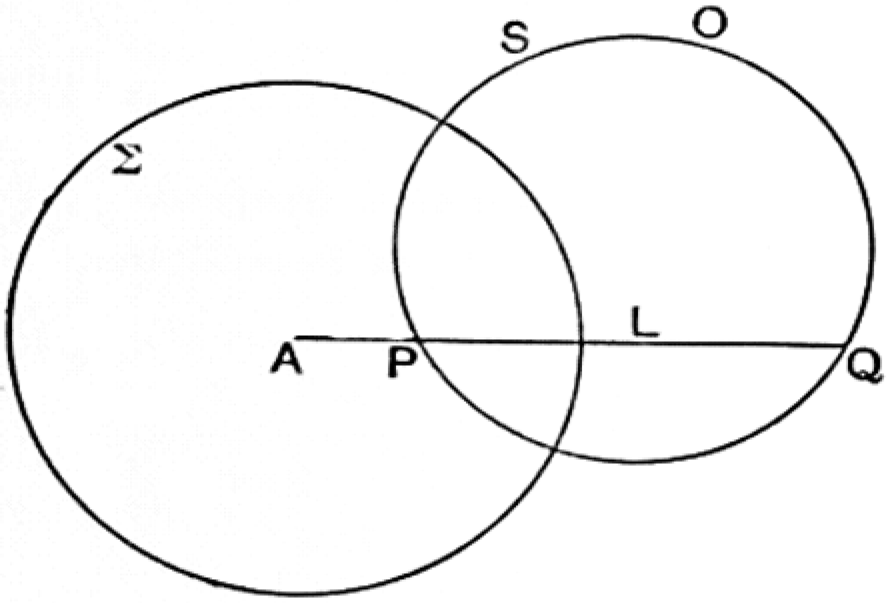

FIG. 70.

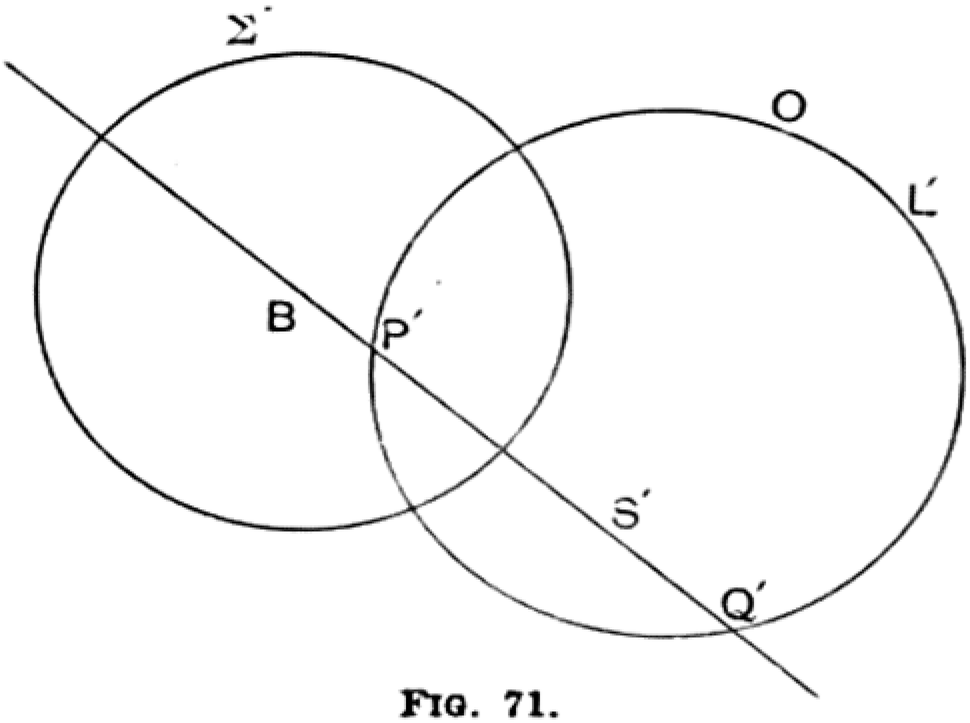

FIG. 71.

(1) Let P, Q be two inverse points w.r.t. circle Σ, centre A.

Take any point O as centre of inversion. Draw a circle through O, P, Q and call it S. Call the line APQ, L.

Since S passes through two inverse points P, Q of Σ it is orthogonal to Σ. [Th. 44.]

When inverted w.r.t. O, Σ becomes a circle Σ' and S becomes a straight line S′ orthogonal to Σ', and $\therefore$ a diameter of Σ'. Also the diameter APQ of Σ becomes a circle L′ through O orthogonal to Σ'.

$\therefore$ the inverse points P′, Q′ of P, Q are the intersections of a diameter of Σ' with a circle orthogonal to Σ', and are therefore inverse points of Σ'. [Th. 44.] Q.E.D.

(2) This follows at once from (1) by regarding C_1 and C_2 as formed of pairs of inverse points P, Q. Q.E.D.

There is one special case which deserves notice.

If in Theorem 55 (1), the point O lies on Σ, then Σ' is a straight line, and it bisects P′Q′ at right angles: in other words, Q′ is the reflection of P′ in Σ'.

Consequently, if in Theorem 55 (2), the point O lies on Σ, C_1' and C_2' are reflections of each other in the line Σ', and are therefore equal curves.

THEOREM 56.

If O is the point of intersection of the exterior common tangents of two circles C_1, C_2, it is possible with O as centre of inversion to invert either circle into the other.

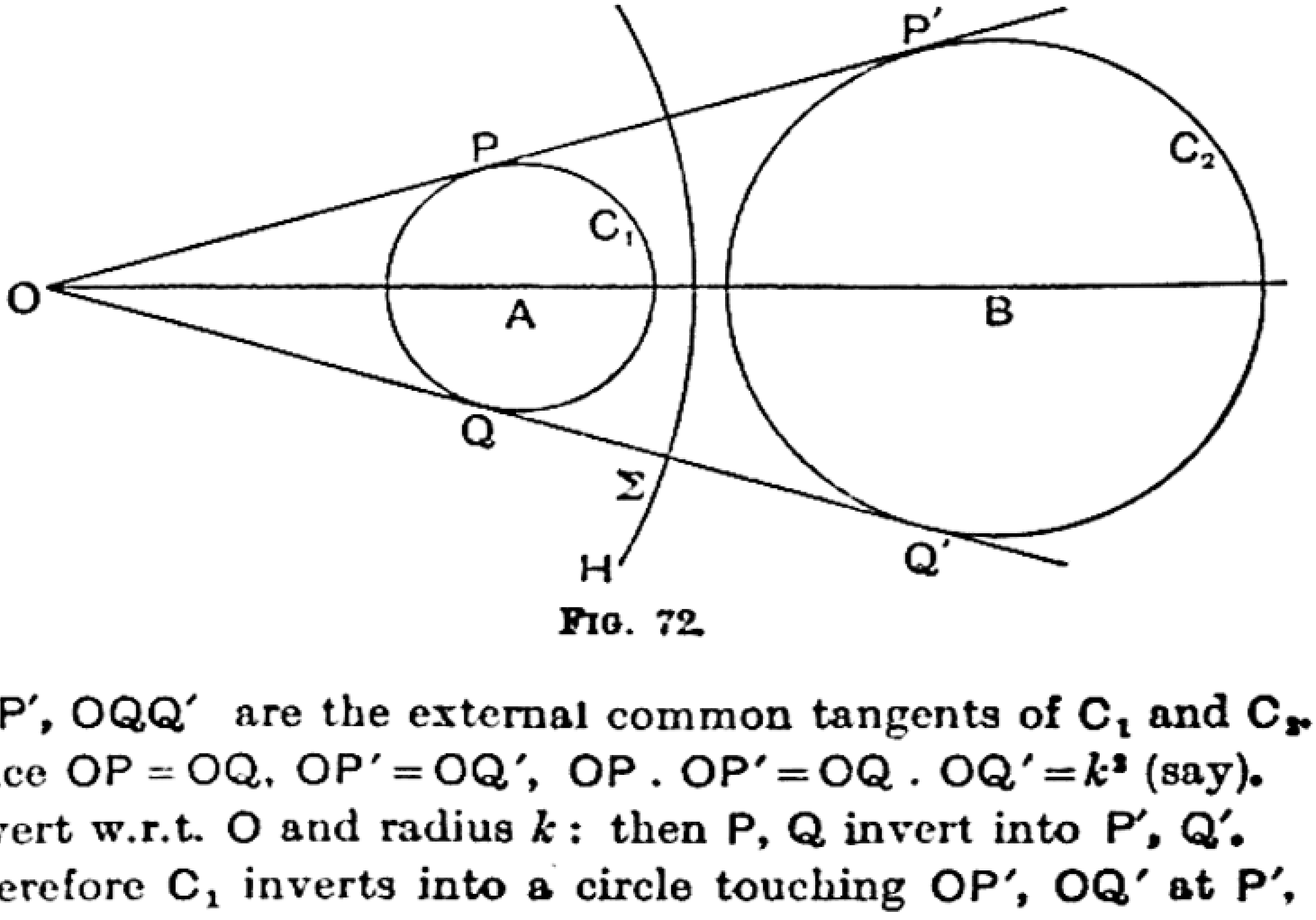

FIG. 72.

OP′, OQQ′ are the external common tangents of C_1 and C_2.

Since OP = OQ, OP′ = OQ′, OP . OP′ = OQ . OQ′ = k^2 (say).

Invert w.r.t. O and radius k: then P, Q invert into P′, Q′.

Therefore C_1 inverts into a circle touching OP′, OQ′ at P′, Q′, and therefore into C_2. Q.E.D.

In other words, C_1 and C_2 are inverse circles w.r.t. the circle, centre O, radius k. Consequently, if H is any point on this circle, and if the system is inverted w.r.t. H, C_1 and C_2 invert into equal circles.

The proof of this theorem also yields another important result.

If through the centre of similitude O of two circles any line ORSS′R′ is drawn, cutting C_1 at R, S and C_2 at S′, R′, then

$$OR \,.\, OR' = OS \,.\, OS' = \text{a constant},$$

for each $= k^2$.

In Theorem 56, the *external* centre of similitude was chosen for O; the process may equally be effected w.r.t. the *internal* centre of similitude, but in this case k^2 is negative.

38. Invert w.r.t. O:

(i) A variable line is drawn through a fixed point O to cut two fixed lines at P, Q; on OP, a point R is taken such that $\frac{1}{OR} = \frac{1}{OP} + \frac{1}{OQ}$, then the locus of R is a straight line.

(ii) If two chords PQ, RS of a circle intersect at A, and if O is a point on the circle, PA . AQ = RA . AS.

39. Two circles intersect at OP; their tangents at O meet the circles again at A, B; if the circle AOB cuts OP produced at Q, prove that OQ = 2OP.

40. [EXTENSION OF PTOLEMY'S THEOREM.] If OABC is a non-cyclic quadrilateral, OA . BC + OC . AB > OB . AC; but if the quadrilateral is cyclic, OA . BC + OC . AB = OB . AC. [Invert w.r.t. O, using Th. 54.]

41. O is a point on the minor arc BC of the circumcircle of the triangle ABC; OP, OQ, OR are the perpendiculars from O to BC, CA, AB; prove that $\frac{BC}{OP} = \frac{BA}{OR} + \frac{AC}{OQ}$. Generalise this for any inscribed polygon.

42. A line AB of constant length moves with its extremities on two fixed lines Ox, Oy; A′, B′ are the inverses of A, B w.r.t. O; prove that the envelope of A′B′ is a circle.

43. A, B, C, D are any four collinear points; A′, B′, C′, D′ are their inverses w.r.t. *any* point O; prove that $\frac{AB \,.\, CD}{AD \,.\, CB} = \frac{A'B' \,.\, C'D'}{A'D' \,.\, C'B'}$.

44. O, A_1, A_2, ... A_n are a system of points on a circle; prove that

$$\frac{A_1A_2}{OA_1 \cdot OA_2} + \frac{A_2A_3}{OA_2 \cdot OA_3} + \dots + \frac{A_{n-1}A_n}{OA_{n-1} \cdot OA_n} = \frac{A_1A_n}{OA_1 \cdot OA_n}.$$

45. Three circles have a common point O; OA, OB, OC are the common chords, and meet the remaining circles at D, E, F; prove that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

46. Enunciate the inverse of Menelaus' theorem.

47. Two circles C_1, C_2 intersect at A, B and cut orthogonally a circle S; prove that A, B are inverse points w.r.t. S.

48. P is the inverse of O w.r.t. a circle Σ; if the system is inverted w.r.t. O, prove that P′ is the centre of Σ'.

49. Two fixed circles cut at O; a variable circle S touches the fixed circles; find the locus of the inverse of O w.r.t. S.

50. Prove that the operations of inversion w.r.t. two circles are commutative if, and only if, the circles are orthogonal.

51. X, X′, and Y, Y′ are pairs of inverse points w.r.t. a circle and lie on a diameter on opposite sides of the centre: P is any point on the circle; prove that $X\hat{P}Y + X'\hat{P}Y' = 180°$.

52. A, B, C are three collinear points; O is a point, such that $A\hat{O}B = B\hat{O}C = 60°$; prove that $\frac{1}{OB} = \frac{1}{OA} + \frac{1}{OC}$.

53. Q, R are the inverses of a point P w.r.t. two fixed circles: if P moves on a coaxal circle, prove that QR passes through a fixed point.

54. Q, Q′ are inverse points w.r.t. a circle C_1; R, R′ are the inverse points of Q, Q′ w.r.t. a circle C_2 orthogonal to C_1; prove that R, R′ are inverse points w.r.t. C_1.

55. C_2 is the inverse of a circle C_1 w.r.t. a circle Σ; another circle touches C_1 and C_2 at P, Q; if both the contacts are internal or both external, prove that P, Q are inverse points w.r.t. Σ.

56. C_1, C_1' and C_2, C_2' are pairs of inverse circles w.r.t. a circle Σ; O is any point on Σ; prove that the inverses of O w.r.t. C_1, C_1', C_2, C_2' are concyclic.

57. P′ is the inverse of P w.r.t. a circle; QP′R is any chord; prove that PP′ bisects $Q\hat{P}R$.

58. If the circles inverse to the given circles ACD, BCD w.r.t. a point P are equal, prove that the circle PCD bisects the angle of intersection of the given circles.

59. Show how to invert three given circles into three equal circles.

60. Four circles A, B, C, D all touch a fifth circle; also A touches B at P, B touches C at Q, C touches D at R; if the circle PQR cuts A, B, C, D orthogonally, prove that a second circle can be drawn to touch A, B, C, D.

61. OABC, OA′B′C′ are two straight lines, such that B, B′ are the mid-points of AC, A′C′; prove that the circles OAA′, OBB′, OCC′ have a common chord.

62. A variable circle touches a fixed circle, and is such that the tangent to it from a fixed point is of constant length; prove that it also touches another fixed circle.

63. A circle touches two circles S_1, S_2 externally at P, Q; prove that PQ passes through the external centre of similitude of S_1, S_2. [Invert so that S_1 turns into S_2.]

64. A circle touches externally two other circles at A, B and meets their common chord at P, Q; prove that AP, BP, AQ, BQ meet the two circles at the points of contact of their common tangents.

65. A point P is inverted successively w.r.t. four circles, each of which is orthogonal to the others; prove that it returns finally to its original position.

66. The extremities of a line PQ of constant length move on two fixed lines OX, OY; PK, QK are drawn perpendicular to OP, OQ; prove that the locus of K is a circle.

67. O is a point of intersection of two orthogonal circles, centres A, B; a line POQ cuts the circles at P, Q; prove that the mid-point of PQ subtends a right angle at AB.

68. A system of circles is drawn to cut a given circle orthogonally at two points of intersection and to pass through a given point *not* in the plane of the circle. Prove that they intersect in another common point.

Hence show how a circle and a point not in its plane can be inverted into a circle and its centre.

69. [**Casey's Theorem.**] t is the length of a common tangent of two circles, radii a, b; t' is the length of the corresponding common tangent of their inverses w.r.t. any point, and a', b' the radii of their inverses; prove that $\dfrac{t^2}{a\,.\,b}=\dfrac{t'^2}{a'\,.\,b'}$.

70. OAA′, OBB′ are two straight lines, such that AB′, A′B intersect on the bisector of the angle AOB; prove that $\dfrac{1}{\text{OA}}-\dfrac{1}{\text{OA}'}=\dfrac{1}{\text{OB}}-\dfrac{1}{\text{OB}'}$.

71. [**Feuerbach's Theorem.**] The nine-point circle of any triangle touches the incircle and each excircle. [With the usual notation, prove that the inverse of the nine-point circle w.r.t. A′ and radius of inversion A′X is the fourth common tangent of the incircle and excircle, centre I_1.]

72. [**Peaucellier's Cell.**] Four equal rods, of length a, are jointed to form the rhombus ABCD; two equal rods OB, OD, of length b $(b>a)$, are jointed at one end O, and their other extremities are jointed to B, D; O is kept fixed. If A traces out any curve, prove that C traces out the inverse curve w.r.t. O and that $k^2=b^2-a^2$.

73. Four equal rods, of length a, are jointed to form the rhombus PQRS; two equal rods OQ, OS, of length b $(a>b)$, are jointed at one end O, and their other extremities are jointed to Q, S; O is kept fixed. If P traces out any curve, prove that R traces out the inverse curve w.r.t. O and that k^2 is negative and $=b^2-a^2$.

If now P is jointed to a fixed point A by a rod AP of length OA, prove that R traces out a straight line, the perpendicular distance of which from O is $\dfrac{a^2-b^2}{2c}$, where $\text{OA}=c$.

Prove also that the length of line which can be traced out by R is

$$2(a+b)\sqrt{\left\{1-\left(\frac{a-b}{2c}\right)^2\right\}}.$$

ANALYTICAL METHODS.

By using polar coordinates, it is easy to apply analysis to Inversion

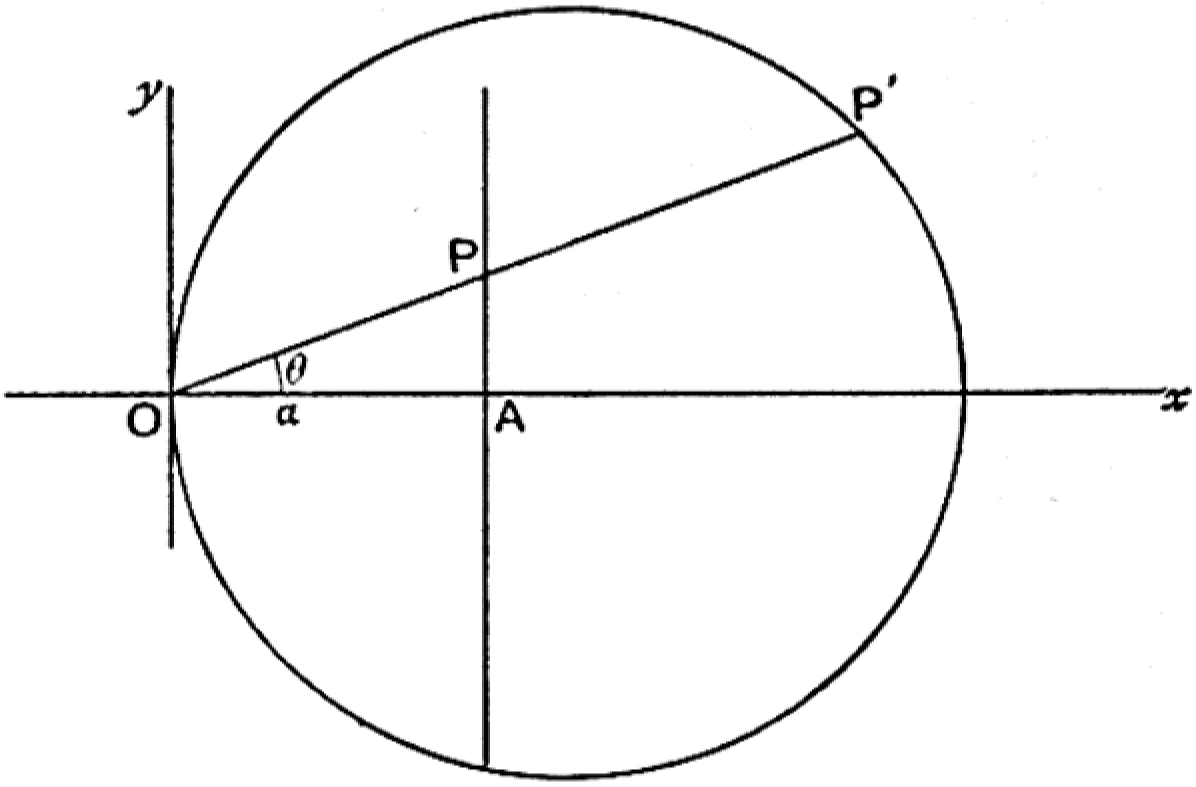

FIG. 73.

Take the origin at the centre of inversion, and let the radius of inversion be k.

Let (r, θ) be the coordinates of any point P and (r', θ') the coordinates of the inverse point P′.

Then $$rr' = k^2 \quad\text{and}\quad \theta = \theta'.$$

$\therefore$ if P traces out the curve $f(r, \theta) = 0$, P′ traces out the curve

$$f\left(\frac{k^2}{r}, \theta\right) = 0.$$

For example, the equation of any circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad\text{or}\quad r^2 + 2r(g\cos\theta + f\sin\theta) + c = 0.$$

$\therefore$ its inverse w.r.t. the origin is

$$\frac{k^4}{r^2} + \frac{2k^2}{r}(g\cos\theta + f\sin\theta) + c = 0,$$

or $$cr^2 + 2k^2r(g\cos\theta + f\sin\theta) + k^4 = 0,$$
which is another circle, unless $c = 0$.

The following list contains a few simple examples of inverse curves, inversion in each case being w.r.t. the origin. The reader should draw freehand the shapes of these various curves.

(i) $r = a\cos\theta$, a circle through the origin: its inverse $r\cos\theta = \frac{k^2}{a}$ is a straight line.

(ii) $\frac{L}{r} = 1 + \cos\theta$, a parabola, origin at the focus: its inverse $r = \frac{k^2}{L}(1 + \cos\theta)$ is a Cardioid, which has a cusp at the pole, *i.e.* the origin.

(iii) $\frac{L}{r} = 1 + e\cos\theta$, a conic, origin at the focus: its inverse $r = \frac{k^2}{L}(1 + e\cos\theta)$ is a Limaçon whose pole is the origin.

(iv) $x^2 - y^2 = a^2$ or $r^2\cos 2\theta = a^2$ is a rectangular hyperbola, origin at the centre · its inverse $r^2 = \frac{k^4}{a^2}\cos 2\theta$ is called a Lemniscate, which has a double point at the pole, *i.e.* the origin.

(v) $y^2 = px$ or $r\sin^2\theta = p\cos\theta$ is a parabola, origin at the vertex: its inverse $r\cos\theta = \frac{k^2}{p}\sin^2\theta$ is called the Cissoid of Diocles, which has a cusp at the pole, *i.e.* the origin.

In the following examples, where a result is to be proved, an elementary knowledge of geometrical conics is required.

74. Invert w.r.t. O: if the tangents at the points P, Q, R of a cardioid, pole O, are parallel, then $\sqrt{OP} \pm \sqrt{OQ} \pm \sqrt{OR} = 0$.

75. Invert w.r.t. S: the circumcircle of the triangle formed by three tangents to a parabola passes through the focus S.

76. Invert w.r.t. S: if PSQ is a variable chord of a parabola, focus S, then $\frac{1}{PS} + \frac{1}{SQ}$ is constant.

77. Invert w.r.t. C: if PCQ is a diameter of a rectangular hyperbola, centre C, and if LR is any chord, then the angles LPR, LQR are equal or supplementary.

78. A circle is drawn through the pole O of a cardioid to touch the curve at P and meets the axis again at T; prove that OP = OT.

79. A variable circle is drawn through the pole O of a cardioid to touch the curve; prove that the locus of the extremity of the diameter through O is a circle.

80. A variable circle is drawn through the pole O of a cardioid to touch the curve; if P is the point of contact, prove that the diameter of the circle varies as $\sqrt{OP}$.

81. A circle is drawn through the pole O of a cardioid to cut the curve orthogonally at P; if it cuts the axis at G, prove that OG = OP.

82. POQ is a variable chord of a Limaçon, pole O; two circles are drawn through O to touch the curve at P, Q respectively: if they meet again at R, prove that the locus of R is a circle.

83. Invert w.r.t. S: a circle of constant radius passes through the focus S of a conic and cuts it at A, B, C, D; then (i) SA . SB . SC . SD is constant; (ii) $\frac{1}{SA}+\frac{1}{SB}+\frac{1}{SC}+\frac{1}{SD}$ is constant.

84. Invert w.r.t. O: (i) the tangents at the ends of a chord through the pole O of a cardioid are at right angles; (ii) the normals at the ends of a variable chord through the pole O of a Limaçon intersect on a fixed circle.

85. Invert w.r.t. C: two rectangular hyperbolas have the same centre C: if the axes of one are the asymptotes of the other, then they cut orthogonally.

86. The cusp of a cissoid is at A, and AX is the tangent to the cusp; a circle is drawn through A to touch the curve at P and cuts AX at Q; a line through P perpendicular to AP meets AX at R; prove that QA = AR.

87. If the curve $f(r, \theta)=0$ is transferred by the substitution $r=R^n$, $\theta=n\Theta$, prove that the angle at which any two curves cut is equal to the angle at which the corresponding curves cut at the corresponding point.

CHAPTER XI.

COAXAL CIRCLES.

Definitions.

(1) If A is the centre of a circle, radius a, and if P is any point, $AP^2 - a^2$ is called the *power* of P w.r.t. the circle.

Note that if P lies outside the circle, and if PH is the tangent from P to the circle, the power of P w.r.t. the circle $= PH^2$.

(2) The locus of a point P which moves so that its powers w.r.t. two circles are equal is called the *radical axis* of the two circles.

Note that points so situated that the tangents from them to the two circles are of equal length lie on the radical axis.

(3) If a system of circles is such that the radical axis of any pair is the same as that of any other pair, the circles are said to form a *coaxal system.*

For example, all circles which pass through two fixed points AB form a coaxal system, and AB is the radical axis.

Many of the properties of coaxal circles admit of simple analytical treatment.

For this purpose, the following abbreviations are useful:

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1,$$

$$S_1(\xi, \eta) \equiv \xi^2 + \eta^2 + 2g_1\xi + 2f_1\eta + c_1,$$

and similarly for S_2, S_3, etc.

The equation of any circle $S_1 = 0$ can be written in the form

$$(x + g_1)^2 + (y + f_1)^2 = g_1^2 + f_1^2 - c_1.$$

This shows that its centre is $(-g_1, -f_1)$ and its radius

$$= \sqrt{(g_1^2 + f_1^2 - c_1)}.$$

I. The length of the tangent from (ξ, η) to $S_1=0$ is $\sqrt{S_1(\xi, \eta)}$.

Let PH be the tangent from P to the circle, centre A, whose equation is $S_1=0$. (See Fig. 74.)

Now $$P \text{ is } (\xi, \eta) \quad \text{and} \quad A \text{ is } (-g_1, -f_1).$$

$$\therefore \ PA^2=(\xi+g_1)^2+(\eta+f_1)^2.$$

$$\therefore \ PH^2=PA^2-AH^2=(\xi+g_1)^2+(\eta+f_1)^2-(g_1^2+f_1^2-c_1)$$
$$=\xi^2+\eta^2+2g_1\xi+2f_1\eta+c_1=S_1(\xi, \eta).$$

$$\therefore \ PH=\sqrt{S_1(\xi, \eta)}.$$

II. The radical axis of two circles is a straight line which passes through the points of intersection (real or imaginary) of the two circles.

Let $S_1=0$ and $S_2=0$ be the two circles: let (ξ, η) be any point on their radical axis.

$\therefore$ by I., $$\sqrt{S_1(\xi, \eta)}=\sqrt{S_2(\xi, \eta)}.$$

$$\therefore \ \xi^2+\eta^2+2g_1\xi+2f_1\eta+c_1=\xi^2+\eta^2+2g_2\xi+2f_2\eta+c_2,$$

or $$2\xi(g_1-g_2)+2\eta(f_1-f_2)+c_1-c_2=0.$$

$\therefore$ the locus of (ξ, η) is a straight line. Q.E.D.

Further, any point whose coordinates satisfy $S_1=0$ and $S_2=0$ must satisfy $S_1=S_2$, which is the radical axis.

$\therefore$ the radical axis passes through the common points of $S_1=0$ and $S_2=0$. Q.E.D.

III. The three radical axes of three circles taken in pairs are concurrent.

Let $S_1=0$, $S_2=0$, $S_3=0$ be the three circles.

Let (ξ, η) be the point of intersection of the radical axis of $S_1=0$, $S_2=0$ with the radical axis of $S_2=0$, $S_3=0$.

$$\therefore \ S_1(\xi, \eta)=S_2(\xi, \eta) \quad \text{and} \quad S_2(\xi, \eta)=S_3(\xi, \eta).$$

$$\therefore \ S_1(\xi, \eta)=S_3(\xi, \eta).$$

$\therefore$ (ξ, η) lies on the radical axis of $S_1=0$, $S_3=0$. Q.E.D.

IV. A point P moves so that the lengths of the tangents from P to two given circles are in a constant ratio λ; then the locus of P is a circle, coaxal with the given circles.

Let $S_1=0$, $S_2=0$ be the given circles, and let (ξ, η) be coordinates of P.

Then $\dfrac{S_1(\xi, \eta)}{S_2(\xi, \eta)} = \lambda^2$ or $S_1(\xi, \eta) - \lambda^2 S_2(\xi, \eta) = 0$.

$\therefore$ the locus of P is $S_1(x, y) - \lambda^2 S_2(x, y) = 0$.

But this is a circle, for the coefficients of x^2 and y^2 are equal and the coefficient of xy is zero.

Further, it is coaxal with $S_1 = 0$ and $S_2 = 0$, for the coordinates of any point which satisfy $S_1 = 0$ *and* $S_2 = 0$ must satisfy $S_1 - \lambda^2 S_2 = 0$. Consequently, $S_1 - \lambda^2 S_2 = 0$ passes through the points of intersection of $S_1 = 0$, $S_2 = 0$, and is therefore by II. coaxal with them.

V. (i) Any circle coaxal with $S_1 = 0$ and $S_2 = 0$ can be written in the form $S_1 + k \,.\, S_2 = 0$, where k is an arbitrary constant.

(ii) If P is a variable point on a given circle coaxal with two given circles $S_1 = 0$, $S_2 = 0$, then the ratio of the tangents from P to $S_1 = 0$ and $S_2 = 0$ is constant.

The proof is left to the reader.

VI. If the axes of reference are chosen so that the y-axis is the radical axis of a system of coaxal circles, and so that the x-axis passes through the centre of one of the circles, then any circle of the system can be written in the form $x^2 + y^2 + 2\lambda x + c = 0$, where λ varies and c is constant.

Let the equation of the one circle whose centre lies on the x-axis be $x^2 + y^2 + 2gx + c = 0$.

Let any other circle of the system be $x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0$.

$\therefore$ the radical axis is $2(g_1 - g)x + 2f_1 y + c_1 - c = 0$.

But the radical axis is $x = 0$; $\therefore$ $f_1 = 0$ and $c_1 = c$.

$\therefore$ any other circle of the system becomes $x^2 + y^2 + 2g_1 x + c = 0$.

Q.E.D.

Since, for different values of λ, $x^2 + y^2 + 2\lambda x + c = 0$ represents different circles of a coaxal system, and since

$$x^2 + y^2 + 2\lambda x + c \equiv (x + \lambda)^2 + y^2 + c - \lambda^2,$$

if we choose $\lambda^2 = c$ or $\lambda = \pm\sqrt{c}$, the two circles $(x \pm \sqrt{c})^2 + y^2 = 0$, so obtained, are members of the coaxal system.

These are circles of zero radius or point-circles. Therefore every coaxal system possesses two point-circles $(+\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$, real or imaginary, which are called the *limiting points* of the system.

VII. The polars of a fixed point P w.r.t. the circles of a coaxal system all pass through the same point Q; and PQ is bisected by the radical axis.

Let (ξ, η) be the coordinates of P, and let $x^2+y^2+2\lambda x+c=0$ be any circle of the system.

The polar of P is $\quad x\xi+y\eta+\lambda(x+\xi)+c=0,$

or $\quad (x\xi+y\eta+c)+\lambda(x+\xi)=0.$

Now, for *all* values of λ, this passes through the point given by $x+\xi=0$, $x\xi+y\eta+c=0$, *i.e.* $\left(-\xi, \dfrac{\xi^2-c}{\eta}\right)$: this then is Q, and the coordinates of the mid-point of PQ are $\left(0, \dfrac{\xi^2-c}{2\eta}+\frac{1}{2}\eta\right)$, which lies on the radical axis.

Q.E.D.

THEOREM 57.

The radical axis of two circles is a straight line perpendicular to the line joining the centres of the circles.

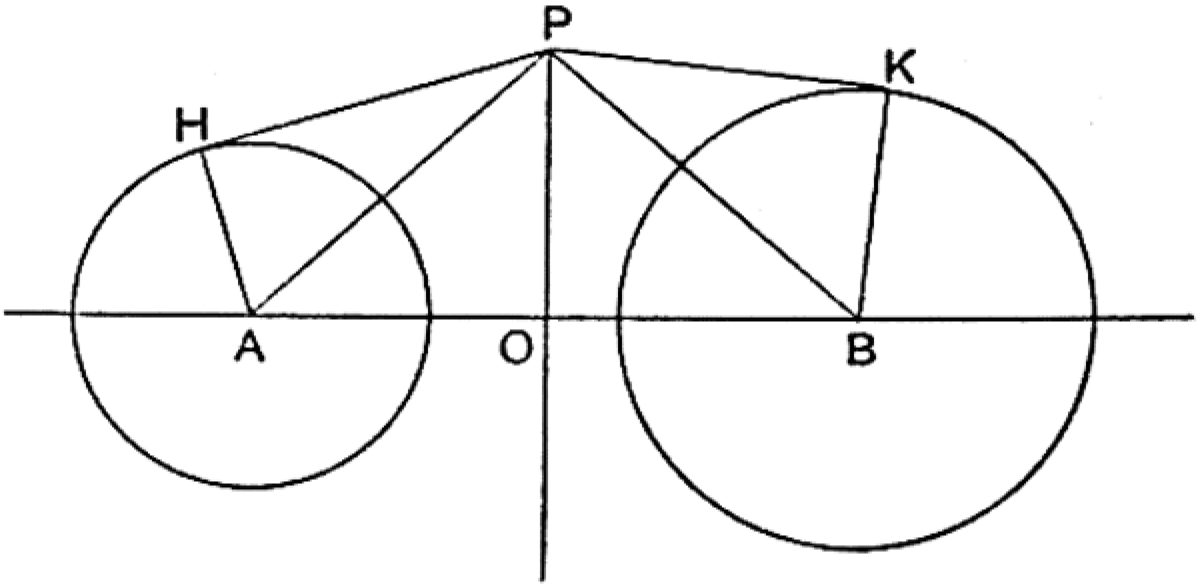

FIG. 74.

A, B are the centres. Let P be any point on the radical axis. Draw PO perpendicular to AB.

$$\therefore \text{PA}^2-\text{AH}^2=\text{PB}^2-\text{BK}^2, \text{ by Definition.}$$

$$\therefore \text{PO}^2+\text{OA}^2-\text{AH}^2=\text{PO}^2+\text{OB}^2-\text{BK}^2.$$

$$\therefore \text{OA}^2-\text{OB}^2=\text{AH}^2-\text{BK}^2=\text{constant.}$$

$$\therefore (\text{AO}+\text{OB})(\text{AO}-\text{OB})=\text{AB}(\text{AO}-\text{OB})=\text{constant.}$$

$$\therefore \text{O is a fixed point.}$$

∴ the locus of P is a line through O perpendicular to AB.

Q.E.D.

If the circles intersect at real points C, D, then P lies on CD, or the common chord is the radical axis.

If a system of circles is coaxal, their centres must be collinear; for if A is the centre of one circle, the centres of all the other circles must lie on the perpendicular from A to the radical axis.

THEOREM 58.

The radical axes of three circles taken in pairs are concurrent.

Let S_1, S_2, S_3 be the circles and L_1, L_2, L_3 the radical axes of the pairs S_2, S_3; S_3, S_1; S_1, S_2.

Either L_1, L_2, L_3 are all parallel, or two of them, say L_1 and L_2, meet at O, say.

Since O lies on L_1, the power of O w.r.t. S_2 equals its power w.r.t. S_3.

Since O lies on L_2, the power of O w.r.t. S_3 equals its power w.r.t. S_1.

$\therefore$ the power of O w.r.t. S_1 equals its power w.r.t. S_2.

$\therefore$ O lies on the radical axis of S_1 and S_2, *i.e.* L_3. Q.E.D.

Definition.

The point of intersection of the three radical axes of three circles, taken in pairs, is called their *radical centre*.

THEOREM 59.

(1) If the line of centres of a coaxal system meets the radical axis at O, and if A is the centre and a the radius of any one circle of the system, then $OA^2 - a^2$ is constant.

(2) If the centres of a system of circles are collinear, and if O is a point on this line such that its powers w.r.t. all the circles are equal, then the system is coaxal.

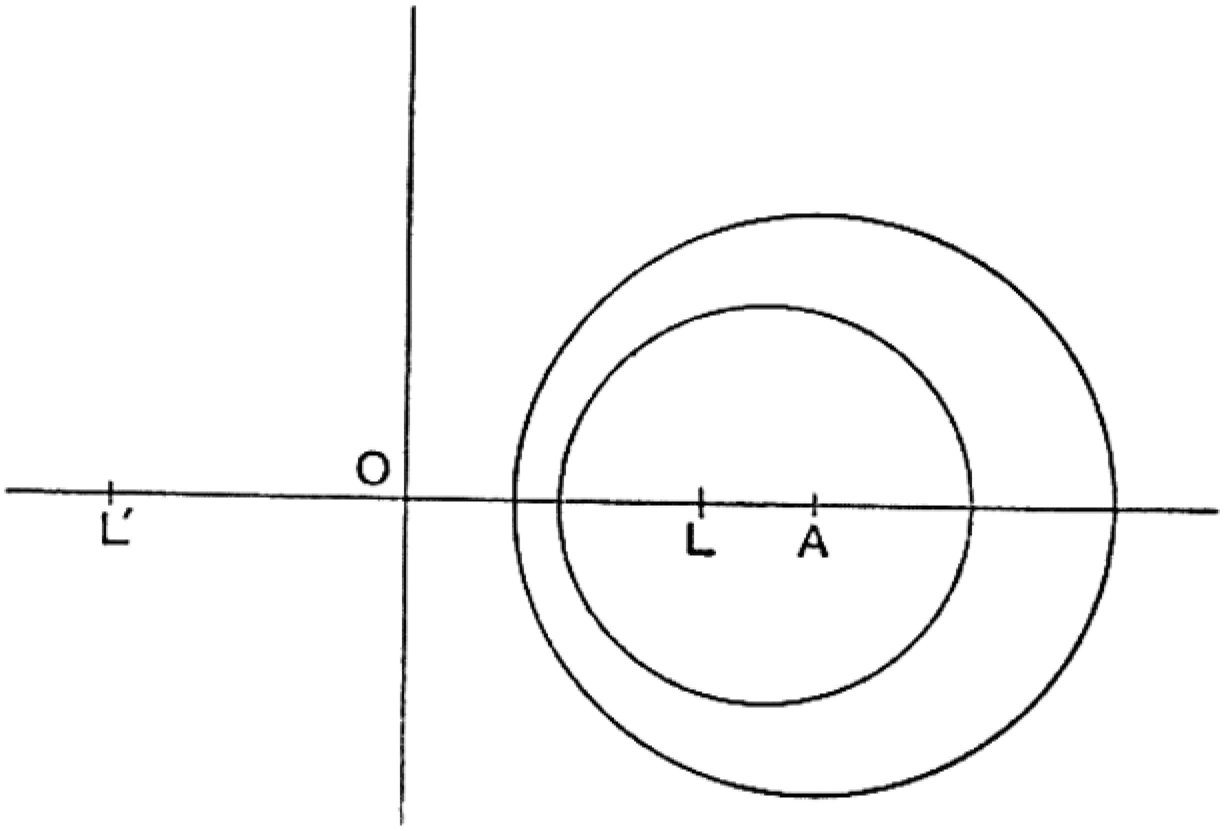

FIG. 75.

(1) Since the circles are coaxal and O is on the radical axis, the powers of O w.r.t. all the circles are equal; but the power of O is $OA^2 - a^2$.

$\therefore$ $OA^2 - a^2$ is constant. Q.E.D.

(2) Since the powers of O w.r.t. any two of the circles are equal, the radical axis of those two circles is the line through O perpendicular to the line of centres.

This is therefore the same line for each pair of circles; therefore the system is coaxal. Q.E.D.

This theorem gives one way of constructing a system of coaxal circles.

Take a line OX, and on it mark any number of points P, Q, R, S, ... ;

take any constant c, and calculate the numbers $p, q, r, s, \ldots$, given by the equations

$$c = \text{OP}^2 - p^2 = \text{OQ}^2 - q^2 = \text{OR}^2 - r^2 = \ldots .$$

Then $p, q, r, \ldots$ are the radii of the circles, centres P, Q, R, ..., which form a coaxal system. The reader is advised to do this as a numerical example, taking some definite value for c.

Two cases arise, resulting in two apparently different types of systems.

If c is positive, no two circles cut each other at real points, and the tangents from O to all circles of the system are equal to $\sqrt{c}$.

If c is negative, every circle of the system cuts every other circle at the same two points, viz. two points E, F on the radical axis, such that $\text{OE}^2 = \text{OF}^2 = -c$.

The best way of understanding this is to draw the two systems corresponding, say, to $c = +9$ and $c = -4$, taking 1 inch as the unit of length.

Consider now the system defined by the equation

$$c = \text{OP}^2 - p^2 = \text{OQ}^2 - q^2 = \ldots ,$$

and suppose c is positive and $= k^2$.

Take two points L, L′ on the line OPQ, such that $\text{L'O} = \text{OL} = k$.

Then the radii l, l' corresponding to the circles, centres L, L′, which belong to the system, are given by $k^2 = \text{OL}^2 - l^2 = \text{OL'}^2 - l'^2$, but $\text{OL}^2 = \text{OL'}^2 = k^2$; $\therefore\ l = 0 = l'$.

$\therefore$ the circles, centres L, L′, which belong to the coaxal system are of zero radius or are point-circles. If c is negative and equal to $-k^2$, we should have $\text{L'O} = \text{OL} = \sqrt{-k^2}$, so that the points L, L′ are imaginary.

Definition.

The two point-circles of a coaxal system are called the *limiting points* of the system.

The tangent from any point to a point-circle is of course simply the line joining the two points. Therefore, if L, L′ are the limiting points, $\text{OL} = \text{OL'} =$ the tangent from O to any other circle of the system.

In the following work, L, L′ will be taken to represent the limiting points, unless otherwise stated.

1. A circle Σ cuts a circle S_1 at A, B and a circle S_2 at C, D; prove that AB, CD intersect on the radical axis of S_1, S_2.

2. What is the radical axis of two circles which touch each other?

3. A circle Σ cuts the coaxal circles S_1, S_2, S_3 at P_1, Q_1; P_2, Q_2; P_3, Q_3; prove that P_1Q_1, P_2Q_2, P_3Q_3 are concurrent.

4. P, Q, R, S are the mid-points of the four common tangents of two circles; prove that P, Q, R, S are collinear.

5. A common tangent PQ of two circles cuts a circle coaxal with them at R, S; prove that {PQ ; RS} is harmonic.

6. If a circle is orthogonal to each of two circles, prove that its centre lies on the radical axis of the two circles and that it passes through their limiting points.

7. P is a point on the radical axis of two circles; prove that the polars of P w.r.t. the circles meet on the radical axis.

8. Construct the radical axis of two non-intersecting circles. [Use Ex. 1.]

9. S_1, S_2, S_3 are three circles: if the centres of S_1, S_2 lie on the radical axes of S_2, S_3 and S_3, S_1 respectively, prove that the centre of S_3 lies on the radical axis of S_1, S_2.

10. Prove that the radical axis of the incircle and one excircle bisects one side of the triangle.

11. A, B, C, D are four circles: if (AB) denotes the radical axis of A and B, etc., and if (AB) is perpendicular to (CD), and if (AC) is perpendicular to (BD), prove that (AD) is perpendicular to (BC).

12. A, B, C are the points of contact of three circles, each of which touches the other two; prove that the tangents at A, B, C are concurrent.

13. P, Q, R are points on BC, CA, AB; prove that the radical centre of the circles whose diameters are AP, BQ, CR is the orthocentre of ABC.

14. Three circles are such that each intersects the other two; prove that the common chords are concurrent.

15. Two circles intersect at P, Q and cut a third circle S orthogonally; prove that P, Q are inverse points w.r.t. S.

16. ABCD is a fixed cyclic quadrilateral: two circles PAB, PCD are drawn to touch at P; prove that the locus of P is a circle.

17. Prove that the locus of the centre of a circle which bisects the circumferences of two given circles is a straight line.

18. Show how to construct a circle orthogonal to each of three given circles.

19. Prove that the circle on LL′ as diameter is orthogonal to every circle of the coaxal system.

20. P is any point on the radical axis, PH is a tangent to a circle of the system; prove that PH = PL.

21. PQ is a common tangent to two circles of a coaxal system; prove that $P\hat{L}Q = 90°$.

22. Prove that the tangent from L to any circle of the coaxal system is bisected by the radical axis.

23. If two coaxal systems have one circle in common, prove that there is one circle orthogonal to all the circles.

24. A variable circle passes through two fixed points A, B and cuts a fixed circle at P, Q; prove that PQ passes through a fixed point.

25. AD, BE, CF are the altitudes of ABC; P is any point; prove that the circles PAD, PBE, PCF are coaxal.

26. ABCD is a cyclic quadrilateral; S is any circle having A, B as limiting points; Σ is any circle having C, D as limiting points; prove that the radical axis of S and Σ passes through a fixed point.

27. Prove that the centroid of three uniform thin rods forming a triangle is the radical centre of the three excircles of the triangle.

THEOREM 60.

(1) If a circle cuts each of two circles orthogonally it passes through their limiting points and is orthogonal to every circle coaxal with them.

(2) The limiting points of a coaxal system are inverse points w.r.t. any circle of the system.

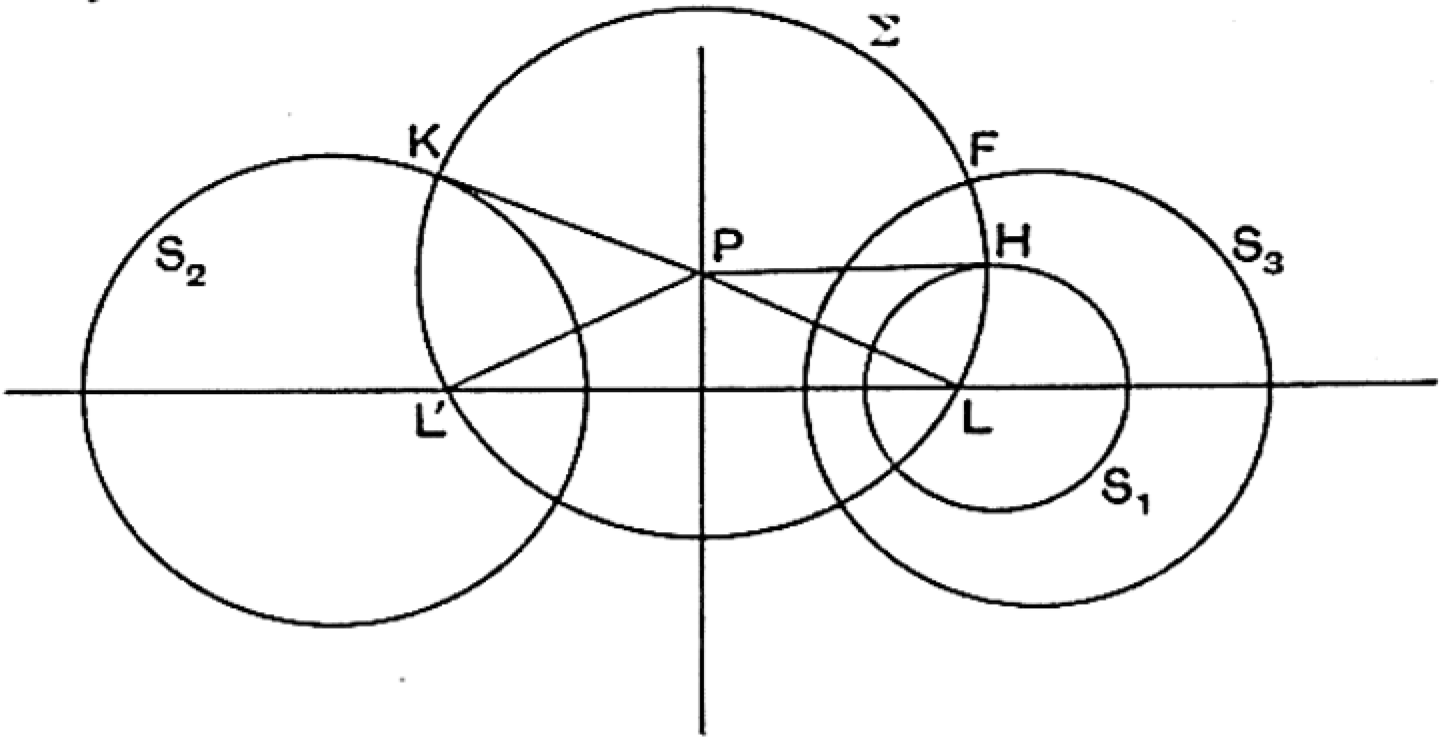

FIG. 76.

(1) Let the circle Σ, centre P, cut circles S_1, S_2 orthogonally at H, K. Then PH, PK are tangents to S_1, S_2 and PH = PK radii;

∴ P lies on the radical axis of S_1 and S_2.

Let L, L′ be the limiting points of S_1, S_2; regarding them as point circles, so that PL, PL′ are tangents, we have PL = PL′ = PH.

∴ L, L′ lie on Σ. Q.E.D.

Let any other circle S_3 of the system cut Σ at F.

Then PF = PH radii: but P lies on the radical axis.

∴ PF is a tangent to S_3.

∴ Σ cuts S_3 orthogonally. Q.E.D.

(2) By Theorem 44, if a circle Σ cuts any circle S_3 orthogonally, it cuts any diameter of S_3 at points inverse to S_3.

∴ L, L′ are inverse points w.r.t. any circle of the coaxal system.

Q.E.D.

THEOREM 61.

The polars of a given point P w.r.t. the circles of a coaxal system are concurrent: and if R is the point of concurrence, PR is bisected by the radical axis.

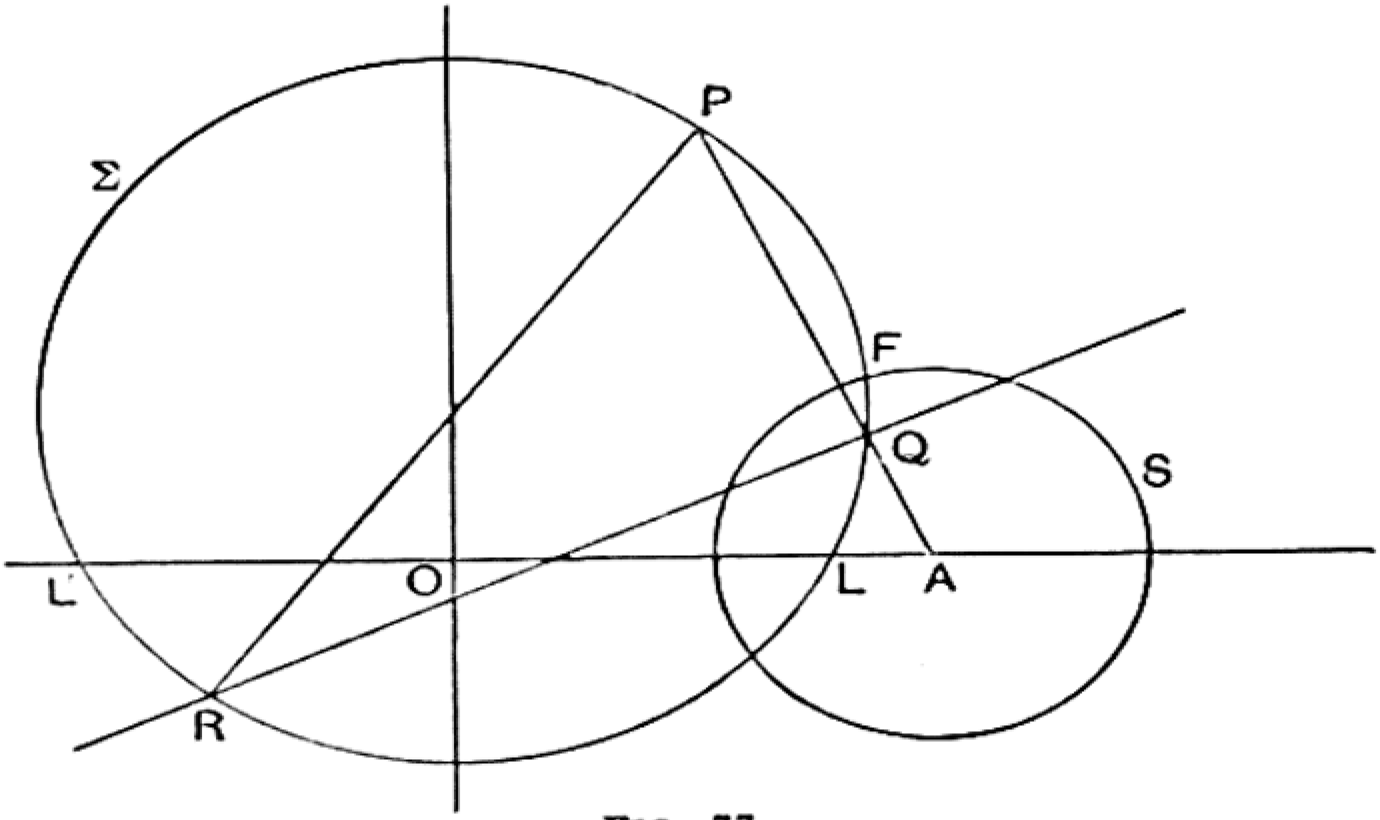

FIG. 77.

Let L, L′ be the limiting points of the coaxal system.

Describe a circle Σ through P, L, L′; then Σ is orthogonal to every circle of the system.

Let A be the centre of any circle S of the system; join AP cutting Σ at Q.

Since S, Σ are orthogonal, P and Q are inverse points w.r.t. S.

Let PR be a diameter of Σ; join RQ.

Since $R\hat{Q}P = 90°$, RP being a diameter, and since P, Q are inverse points, ∴ RQ is the polar of P w.r.t. S.

∴ the polar of P w.r.t. S passes through the fixed point R. Q.E.D.

Further, since Σ passes through L, L′, its centre lies on the perpendicular bisector of LL′, *i.e.* the radical axis.

∴ the mid-point of PR lies on the radical axis. Q.E.D.

It is important to become accustomed to the idea of regarding the limiting points as point-circles. Readiness in recognising this fact is frequently of use in rider-work.

The following method of establishing the fundamental harmonic pole and polar property emphasises this idea. For further examples, see Ex. 38-42.

Example.

A variable line through a fixed point P meets a given circle Σ, centre O, at H, K; Q is a point on HK, such that {HK; PQ} is harmonic; then the locus of Q is a straight line perpendicular to OP.

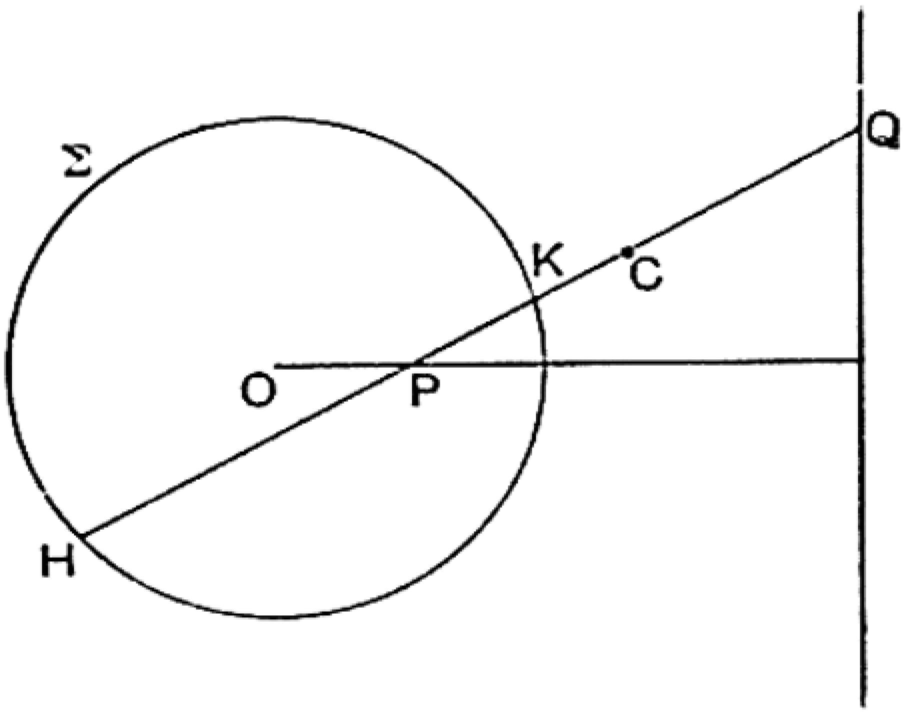

FIG. 78.

Bisect PQ at C: since {HK; PQ} is harmonic, CH . CK $=$ CP^2.

$\therefore$ C is a point on the radical axis of Σ and the point-circle P.

$\therefore$ the locus of C is a straight line perpendicular to OP.

But P is a fixed point and PC $=$ CQ.

$\therefore$ the locus of Q is a straight line perpendicular to OP. Q.E.D.

28. Show how to draw a circle coaxal with two given circles and to touch a given line.

29. A straight line ABCD meets one circle at A, B and another at C, D; if L is a limiting point, prove that $A\hat{L}D + B\hat{L}C = 180°$.

30. A chord PQ of one circle touches a second circle at R; if L is a limiting point, prove that LR bisects $P\hat{L}Q$.

31. If two points are inverse points w.r.t. each of a system of circles, prove that the system is coaxal and has these two points as limiting points.

32. Prove that the three pairs of limiting points of three circles taken in pairs are concyclic.

33. AB is a common tangent of two non-intersecting circles; prove that the circle on AB as diameter passes through their limiting points.

34. Prove that the four circles whose diameters are the four common tangents of two non-intersecting circles are coaxal.

35. A variable circle passes through a fixed point and cuts a given circle orthogonally; prove that it belongs to a fixed coaxal system.

36. A sphere is inverted w.r.t. any point on its surface: determine the inverses of the parallels of longitude and latitude.

37. From a limiting point L, a tangent LP is drawn to a circle of the coaxal system and cuts another circle of the system at H, K; prove {LP ; HK} is harmonic.

38. H, K are the mid-points of the tangents from a point T to a circle Σ; P is any point on HK; prove that the tangent PQ from P to Σ equals PT.

39. OP, OP′ are the tangents from O to a circle; T is any point on the line RR′ bisecting OP, OP′; if the polar of T meets RR′ at Q, prove that $T\hat{O}Q = 90°$.

40. QR is the tangent at Q to a circle; P is any other point: PQ is joined and PR is drawn perpendicular to PQ: the rectangle QPRS is completed; prove that the polar of P passes through S. [Regard P as a point-circle.]

41. From two fixed points A, B, tangents AP, AQ, BL, BM are drawn to a variable circle: if P′, Q′, L′, M′ are the mid-points of these four lines, prove that the locus of the meet of P′Q′ and L′M′ is a straight line.

42. P, Q are conjugate points w.r.t. a circle; if PQ subtends a right angle at a point R, prove that the bisector of the tangents from R to the circle bisects PQ.

43. Find the locus of a point which moves so that its polars w.r.t. three given circles are concurrent.

44. A variable circle touches a given line at a given point; prove that the polar of any other fixed point passes through a fixed point.

45. PQ is a common tangent to two circles; prove that P, Q are conjugate points w.r.t. any circle coaxal with them.

46. A quadrilateral is inscribed in one circle and circumscribed about another; prove that the point of intersection of the diagonals is a limiting point of the two circles.

47. The incircle of the triangle ABC and the circle escribed to BC touch BC at X, X_1 and CA at Y, Y_1; prove that the meets of the circles on XX_1 and YY_1 as diameters lie on the line bisecting $B\hat{A}C$.

48. An exterior common tangent of two circles cuts their radical axis at P; L, L′ are the limiting points; prove that PL, PL′ are parallel to the internal common tangents.

49. ABCD is a given square; E is a point on AD produced, such that $\frac{BE}{CE}$ has its greatest value; prove that $AD^2 = EA \cdot ED$.

THEOREM 62.

(1) The orthocentres of the four triangles formed by four straight lines are collinear.

(2) The mid-points of the three diagonals of a complete quadrilateral are collinear.

(3) The circles whose diameters are the three diagonals of a complete quadrilateral, are coaxal and are cut orthogonally by the polar circles of the four triangles formed by the sides of the quadrilateral.

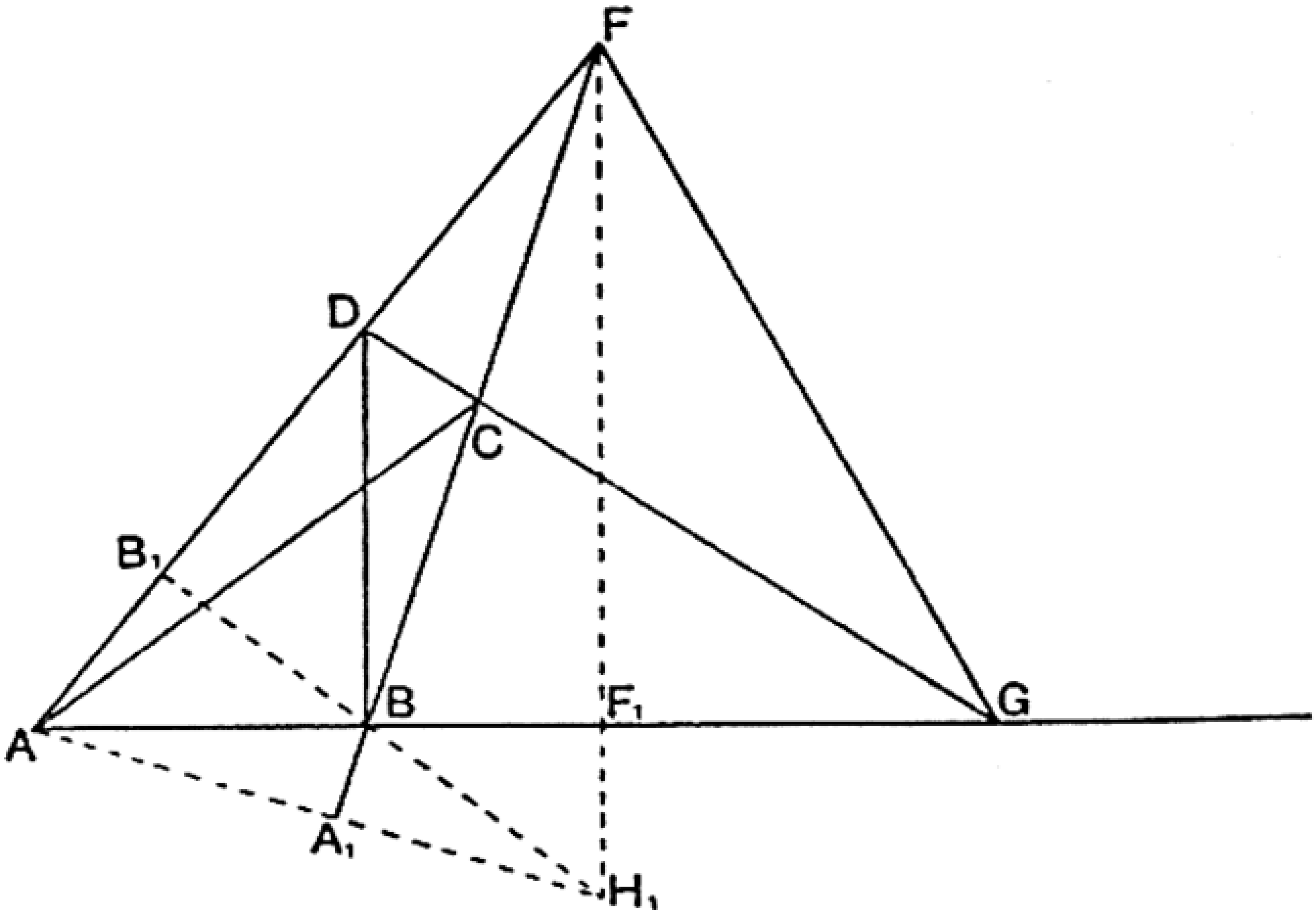

FIG. 79.

Let FG be the third diagonal of the quadrilateral ABCD.

Let H_1 be the orthocentre of the triangle ABF, and let AH_1, BH_1, FH_1 meet BF, FA, AB at A_1, B_1, F_1.

Since H_1 is the orthocentre,

$$H_1A \,.\, H_1A_1 = H_1B \,.\, H_1B_1 = H_1F \,.\, H_1F_1 = r_1^2 \text{ (say).}$$

But AA_1, BB_1, FF_1 are chords of the circle whose diameters are AC, BD, FG; ∴ the tangents from H_1 to these three circles are equal.

If H_2, H_3, H_4 are the orthocentres of ADG, DCF, CBG, in exactly the same way, the tangents from H_2, H_3, H_4 to these circles are equal.

∴ these three circles are coaxal and H_1, H_2, H_3, H_4, being on their radical axis, are collinear: and the mid-points of AC, BD, FG, being their centres, are collinear.

Finally, the circle, centre H_1, radius r_1 is orthogonal to each of these circles; but this is the polar circle of ABF.

∴ the four polar circles are orthogonal to the circles on AC, BD, FG as diameters. Q.E.D.

The reader is reminded that the circumcircle of the diagonal line triangle is also orthogonal to these circles.

It should be noted that the polar circles are not all real; for the polar circle is real, only if the triangle is obtuse-angled.

Definition.

If O, O′ are the centres of similitude of two circles, the circle whose diameter is OO′ is called the *circle of similitude* of the two circles. [See **Ex. 66.**]

THEOREM 63.

(1) The six centres of similitude of three circles, taken in pairs, lie in sets of three points on four straight lines which form a quadrilateral whose diagonal line triangle is the triangle formed by joining the centres of the circles.

(2) The three circles of similitude of three circles taken in pairs are coaxal.

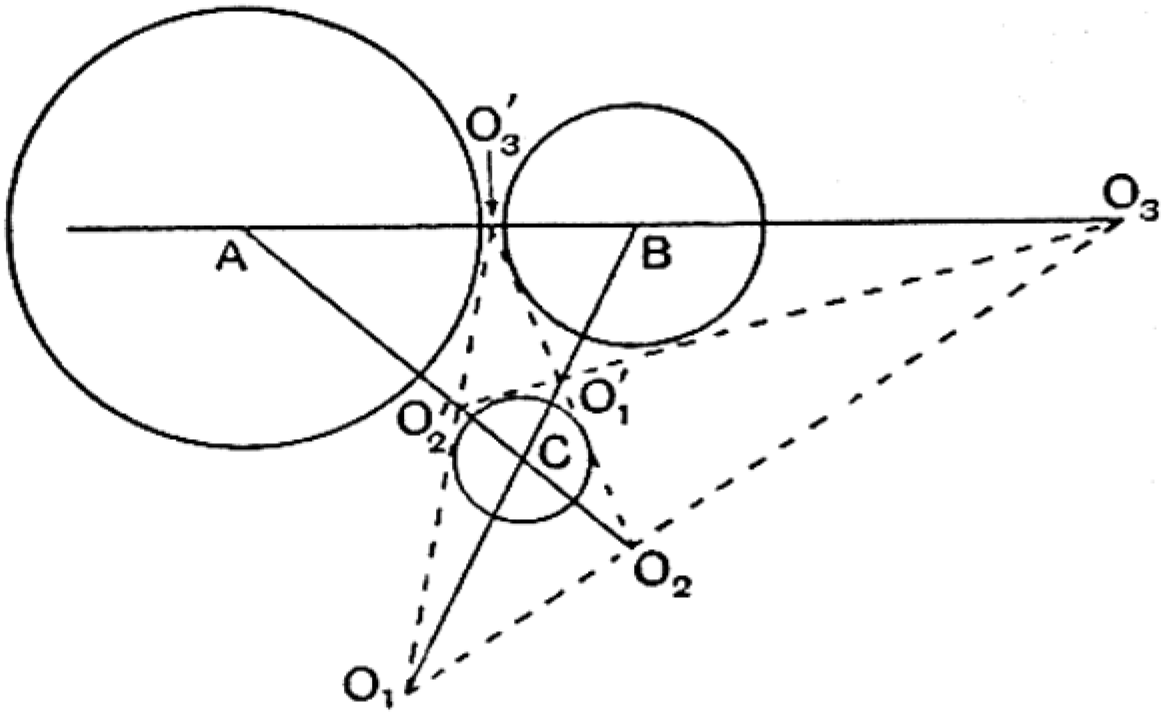

FIG. 80.

(1) A, B, C are the centres of the three circles; O_1, O_1'; O_2, O_2'; O_3, O_3' are their centres of similitude.

Let a, b, c be the radii of the circles, centres A, B, C.

Then
$$\frac{AO_3}{O_3B} = -\frac{a}{b}; \quad \frac{BO_1'}{O_1'C} = \frac{b}{c}; \quad \frac{CO_2'}{O_2'A} = \frac{c}{a}.$$

$$\therefore \frac{AO_3}{O_3B} \cdot \frac{BO_1'}{O_1'C} \cdot \frac{CO_2'}{O_2'A} = -1.$$

$\therefore$ applying Menelaus to triangle ABC, O_3, O_1', O_2' are collinear.

Similarly O_3, O_2, O_1; O_3', O_1', O_2; O_3', O_2', O_1 are sets of collinear points. Q.E.D.

(2) But O_1O_1', O_2O_2', O_3O_3' are the diagonals of a complete quadrilateral; therefore the circles on these lines as diameters are coaxal. Q.E.D.

Corollary.

If a circle S touches two circles Σ_1, Σ_2 at P, Q, then PQ passes through a centre of similitude of Σ_1 and Σ_2.

For P is a centre of similitude of S and Σ_1, and Q is a centre of similitude of S and Σ_2.

50. In a trapezium ABCD, AB and CD are parallel : AD meets BC at E ; prove that the common chord of the circles on AC and BD as diameters passes through E and is perpendicular to AB.

51. Prove that the circumcentre of the diagonal line triangle of a complete quadrilateral is collinear with the orthocentres of the triangles formed by the sides of the quadrilateral.

52. "If a circle is inscribed in a quadrilateral ABCD, its centre lies on the line joining the mid-points of the diagonals." What does this theorem become when A, D, C are collinear ?

53. ABC, PQR are two self-conjugate triangles w.r.t. a circle Σ ; prove that the centre of Σ lies on the radical axis of the circles ABC, PQR.

54. If two circles touch each of two other circles so that 4 or 2 or none of the points of contact are external, prove that the radical axis of either pair passes through a centre of similitude of the other pair.

55. A, B, C are the centres of three circles ; prove that the radical axis of their three circles of similitude passes through the circumcentre of ABC.

56. The tangents at A, B, C to the circumcircle of the triangle ABC cut BC, CA, AB at P, Q, R ; U, V, W are the mid-points of AP, BQ, CR ; prove that UVW is the radical axis of the circumcircle and the polar circle of the triangle ABC.

57. If each of two pairs of opposite vertices of a complete quadrilateral is conjugate w.r.t. a circle, prove that the third pair is also conjugate w.r.t. the circle.

THEOREM 64.

(1) PH, PK are the tangents from a point P to two circles, centres A, B; PN is the perpendicular to their radical axis; then

$$PK^2 - PH^2 = 2AB \cdot PN.$$

(2) From a point P on a circle, centre C, a tangent PK is drawn to a circle, centre B; PN is the perpendicular to their radical axis; then $PK^2 = 2CB \cdot PN$.

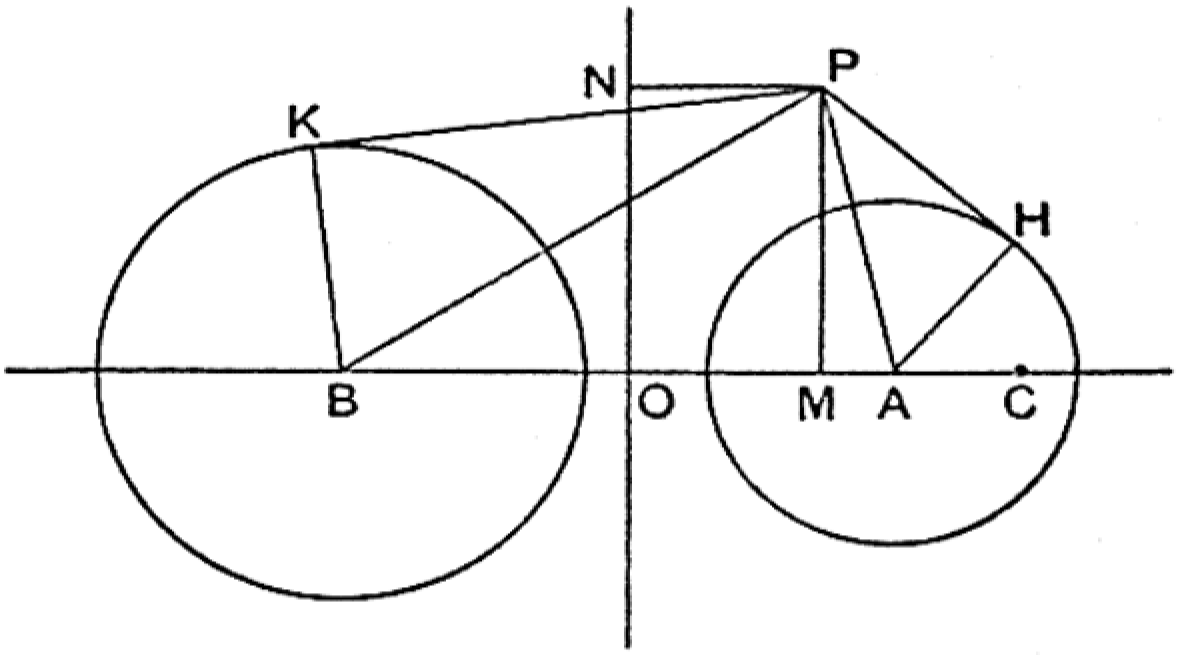

FIG. 81.

(1) Let the radical axis meet AB at O: draw PM perpendicular to AB.

$$\begin{aligned} \text{Then}\quad PK^2 - PH^2 &= (PB^2 - BK^2) - (PA^2 - AH^2) \\ &= PM^2 + MB^2 - PM^2 - MA^2 - (BK^2 - AH^2). \end{aligned}$$

$$\text{But}\quad OB^2 - BK^2 = OA^2 - AH^2 \quad\text{or}\quad BK^2 - AH^2 = OB^2 - OA^2.$$

$$\begin{aligned} \therefore\ PK^2 - PH^2 &= (MB^2 - MA^2) - (OB^2 - OA^2) \\ &= (BM + MA)(BM - MA) - (BO + OA)(BO - OA) \\ &= BA(BM - MA - BO + OA) = BA \cdot 2OM \\ &= 2AB \cdot PN. \end{aligned}$$

Q.E.D.

(2) The tangent from P to the circle, centre C, is of zero length.

$\therefore$ by (1), $PK^2 = 2CB \cdot PN$. Q.E.D.

This theorem admits of an easy analytical proof.

THEOREM 65.

(1) If three circles are coaxal, the ratio of the tangents from a variable point on one of them to the other two circles is constant.

(2) The locus of a point, which moves so that the ratio of the tangents from it to two fixed circles is constant, is a circle coaxal with the given circles.

(1) Let P be a variable point on a circle, centre C, and let A, B be the centres of the other circles: draw PN perpendicular to the radical axis, and draw the tangents PH, PK. (See Fig. 81.)

Then, by Theorem 64 (2),

$$PH^2 = 2PN \, . \, CA \quad \text{and} \quad PK^2 = 2PN \, . \, CB.$$

$$\therefore \ \frac{PH}{PK} \text{ is constant.} \qquad \text{Q.E.D.}$$

(2) Take any one position P of the moving point, and draw through it a circle coaxal with the given circles: let C be its centre.

Then, as in (1), $\dfrac{PH^2}{PK^2} = \dfrac{CA}{AB}$.

$\therefore \ \dfrac{CA}{CB}$ is constant, and therefore C is a fixed point.

$\therefore$ every position of P lies on the same coaxal circle. Q.E.D

THEOREM 66.

The circle of similitude of two circles S_1 and S_2 is coaxal with S_1 and S_2.

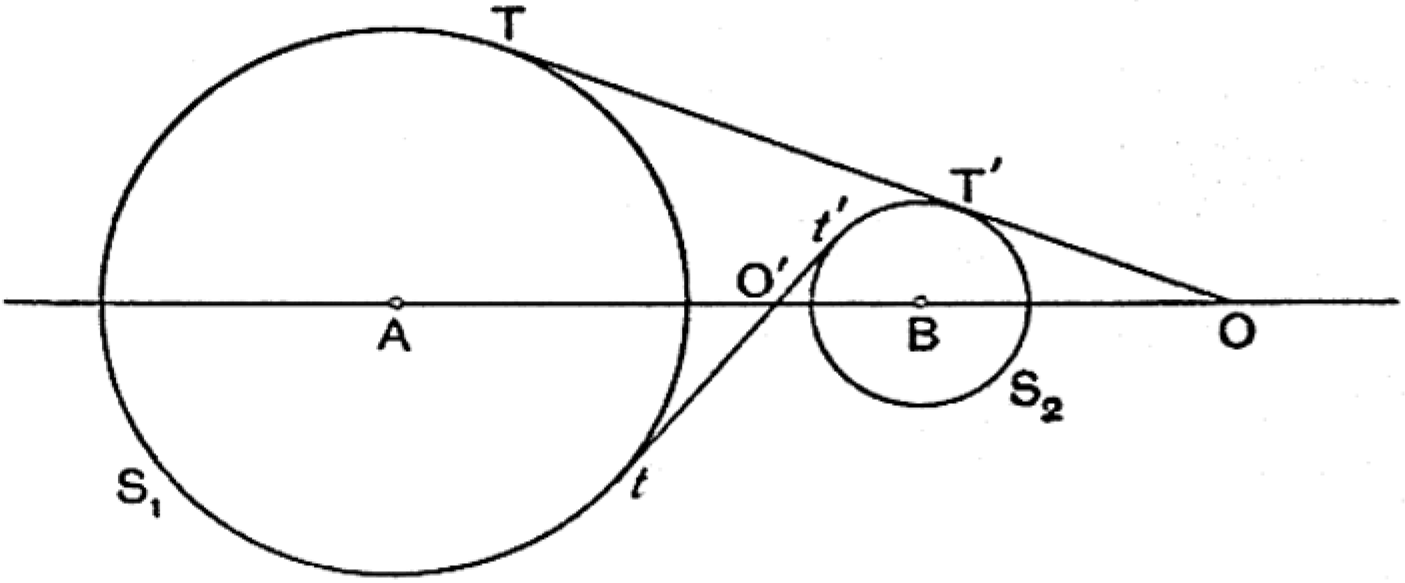

FIG. 82.

Let OT′T and $tO't'$ be two common tangents through the centres of similitude O, O′. Let a, b be the radii of S_1 and S_2.

Then
$$\frac{OT}{OT'} = \frac{a}{b} = \frac{O't}{O't'}.$$

If then a circle is drawn through O′ coaxal with S_1, S_2 it will pass through O, by Theorem 65. But its centre must lie on AO′BO, since it is coaxal with S_1, S_2, and therefore it is the circle of similitude of S_1, S_2. Q.E.D.

58. Prove that the locus of a point, which moves so that the difference of the squares of the tangents from it to two given circles is constant, is a straight line.

59. L is a limiting point of a coaxal system : P is a variable point on a fixed circle of the system; prove that PL^2 varies as the distance of P from the radical axis.

60. S_1, S_2, S_3, S_4 are four coaxal circles; PH_1, PH_2, PH_3, PH_4 are the tangents to them from a variable point P; if $\frac{PH_1 . PH_2}{PH_3 . PH_4}$ is constant, prove that the locus of P consists of two circles.

61. Find the locus of a point which moves so that its distance from a fixed point is proportional to the tangent from it to a fixed circle.

62. A circle Σ touches two given circles and cuts their radical axis at Q, Q′; prove that the tangents at Q, Q′ are parallel to a pair of common tangents of the given circles. [Use Theorem 64, taking P at the centre of Σ.]

63. From any point P, three tangents PX, PY, PZ are drawn to three coaxal circles, centres A, B, C; prove that

$$PX^2 . BC + PY^2 . CA + PZ^2 . AB = 0.$$

64. A is a fixed point: a variable chord PQ of a fixed circle Σ subtends a right angle at A; prove that the locus of the mid-point of PQ is a circle coaxal with Σ and the point-circle A.

65. A straight line cuts one circle at A, B and another circle at C, D; the tangent at A meets the tangents at C, D at P, Q; the tangent at B meets the tangents at C, D at R, S; prove that P, Q, R, S lie on a circle coaxal with the given circles.

66. Prove that two circles subtend equal angles at any point on their circle of similitude.

67. The internal bisector of the angle BAC meets BC at D; prove that the circle on AD as diameter is coaxal with the incircle of the triangle ABC and the circle escribed to BC.

68. Prove that the circle whose diameter is the third diagonal of a cyclic quadrilateral is the circle of similitude of the two circles whose diameters are the other diagonals.

69. PLQ is a variable chord of a fixed circle of a given coaxal system: L is a limiting point; PH, QK are the perpendiculars from P, Q to the radical axis; prove that PH . QK is constant.

70. O is a fixed point; POQ, ROS are two variable perpendicular chords of a fixed circle: prove that the poles of PR, PS, QR, QS lie on a fixed circle coaxal with the given circle and the point-circle O.

71. A, B are the points of contact of a common tangent of two given circles: any line parallel to AB cuts one circle at P and the other at Q. Prove that AP, BQ intersect on a fixed circle, coaxal with the given circles.

THEOREM 67.

A system of coaxal circles can be inverted, either into a system of concurrent straight lines or into a system of concentric circles.

First, suppose the circles intersect at two real points A, B. Invert w.r.t. A. Then each circle becomes a straight line passing through the inverse B′ of B. Q.E.D.

Secondly, suppose the circles do not intersect at real points. In this case, the system has two real limiting points, M, N say.

Draw two circles Σ_1, Σ_2, each passing through M, N. Let C be any one circle of the coaxal system. Invert w.r.t. M; then Σ_1, Σ_2 become two straight lines intersecting at N′, the inverse of N. But C is orthogonal to Σ_1 and Σ_2; therefore its inverse C′ is a circle orthogonal to two lines intersecting at N′.

∴ N′ is the centre of the circle C′.

∴ all the circles become circles having N′ as centre. Q.E.D.

Note that, in particular, any two circles can be inverted either into two straight lines or into two concentric circles.

THEOREM 68.

If a circle S_1 is the inverse of a circle S_2 w.r.t. a circle Σ, then S_1, S_2, Σ are coaxal.

Any point on Σ is unaltered by inversion w.r.t. Σ.

∴ the two points of intersection (real or imaginary) of S_1 and Σ are unaltered by inversion and therefore lie on S_2.

∴ the three circles are coaxal. Q.E.D.

In the case where S_1 and Σ do not meet at real points, another proof may be obtained by constructing two circles orthogonal to S_1 and Σ.

Note that a system of concentric circles may be regarded as a coaxal system, the radical axis being the line at infinity, since the inverse w.r.t. any point is a coaxal system.

72. X is a variable circle touching two fixed circles A, B; P, Q are the two circles which touch X and cut both A and B orthogonally; prove that P, Q cut at a constant angle.

73. Find the locus of the inverse of a fixed point w.r.t. the circles of a coaxal system.

74. S_1, S_2 are two circles touching externally two circles Σ_1, Σ_2; prove that a circle can be drawn coaxal with Σ_1, Σ_2 to cut S_1, S_2 orthogonally.

75. A variable circle touches each of two fixed circles; prove that it cuts any fixed circle coaxal with them at a constant angle.

76. If r is the radius of a variable circle which touches two fixed circles, and if L is one of their limiting points, and if LP is the tangent from L to the variable circle, prove that $\frac{LP^2}{r}$ is constant.

77. Prove that the circumcircle, the nine-point circle and the polar circle of a triangle are coaxal.

78. A, B, C, D are collinear points: S_1, S_2 are the circles whose diameters are AB, CD: a circle orthogonal to S_1 and S_2 cuts S_1 at P, Q and S_2 at H, K; prove that AP, BQ, CH, DK are concurrent.

79. Prove that, if a variable circle is drawn to touch two fixed circles, the ratio of the tangents drawn from the limiting points of the fixed circles to the variable circle is constant.

80. (i) Prove that any three circles can be inverted into three equal circles.

(ii) A circle cuts three given circles at equal angles; prove that it is coaxal with two of the circles which touch all three given circles.

81. Three circles have two common points A, B; a variable line through A cuts the circles at P, Q, R; prove that $\frac{PQ}{QR}$ is constant.

82. L is a limiting point and S_1, S_2 two fixed circles of a coaxal system: Σ is a variable circle touching S_1, S_2; find the locus of the inverse of L w.r.t. Σ.

83. A, B, C, D are four collinear points: T is a point on the radical axis of the circles whose diameters are AB and CD; TA, TB meet the first circle at P, Q, and TC, TD meet the second at R, S; prove that T, P, Q, R, S are concyclic.

84. If two circles are inverted w.r.t. one of their centres of similitude, prove that their radical axis inverts into the circle of similitude of the inverse circles

INDEX.

Apollonius, circle, 16, 68.
  triangle theorem, 61.

Casey's theorem, 117.
Centre of gravity, 51, 58, 59, 62.
Centre of similitude, 1, 3, 29.
Centroid, 21, 52, 62.
Ceva's theorem, 40.
Circle of similitude, 135, 136.
Circumcentre, 19, 20, 28.
Circumradius, 11, 19, 35.
Class, 78.
Coaxal circles, 121, 122, 126, 140.
Complex number, 54.
Conjugate points, 94.
  lines, 96.
  triangles, 97.
Cross-ratio, 73, 75, 100.

Degree, 78.
Demoivre's theorem, 57.
Desargues, 44, 76.
Diagonal triangles, 80, 81, 102, 103.
Duality, 77-81.

Envelope, 77.
Equicross, 74.
Euler line, 28.
Excentre, 23.
Exradius, 25.

Feuerbach's theorem, 117.

Harmonic conjugates, 65.
  pencils, 67.
  properties of orthogonal circles, 91.
  properties of polars, 98.
  system of points and tangents, 70.
  theory of quadrangle, quadrilateral, 102.
Hillyer, 85.
Homothetic, 1, 2.

Incentre, 23.
Infinity, 38, 69.
Inradius, 25.
Inversion, angle unaltered by, 108.
  circle of, 105.
  coaxal circles, 142.
  inverse curves, 113, 114.
  metrical properties, 112.
  radius of, 105.

Join, 77.

Limiting points, 123, 127, 132.
Locus, 77.

Median, 20.
Meet, 77.
Menelaus, 42.

Nine-point circle, 27, 28.
Notation of triangle, 19.

Orthocentre, 20. 28.
Orthogonal circles, 88, **130**.

Pantograph, **5**.
Pascal, 44.
Peaucellier's cell, **117**.
Pedal line, 46.
Pedal triangle, 20.
Pencil, 67.
Polar circle, 97.
Poles and polars, 93, **131**.
Power of point, 121.
Ptolemy's theorem, 17.

Quadrangle, Quadrilateral, 77, 80, 134.

Radical axis, 121, 122, **124**.
Radical centre, **125**.
Range, 73.

Ratio of lengths, 53, 54.
  triangles, 10.

Salmon's theorem, 94.
Self-conjugate, self-polar triangle 97.
Sense, 1, 37.
Similarity, 1.
Simson line, 46.
Species, 6.
Statical methods, 58, 59, **86**

Transversal, 13.
Triangle notation, 19.
Trilateral, 78.

Vector addition, 49.
  multiplication, **53**

w.r.t., 2

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